23. Our differential equation has two different expressions depending on whether \( t < 1 \) or \( t \geq 1 \). Substituting \( 2t \) or \( 2 \) for \( V(t) \) depending on the value of \( t \), we have

\[
\frac{dv_c}{dt} = \begin{cases} 
(2t - v_c)/0.2 = 5(2t - v_c) & \text{for } t < 1; \\
(2 - v_c)/0.2 = 10 - 5v_c & \text{for } t \geq 1;
\end{cases}
\]

(e) For \( t < 1 \), the graph of the solution is identical to the one for \( t < 1 \) in Exercise 21. For \( t \geq 1 \), the graph of the solution is qualitatively similar to the one for \( t \geq 1 \) in Exercise 22. The complete graph is the "combination" of the graphs from Exercises 21 and 22.

EXERCISES FOR SECTION 1.4

1.

<table>
<thead>
<tr>
<th>( k )</th>
<th>( t_k )</th>
<th>( y_k )</th>
<th>( m_k )</th>
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</table>

2. Table 1.2

Results of Euler's method (to two decimal places)

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<tr>
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<th>( y_k )</th>
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</table>

3. Graph for Table 1.2

Graph showing \( y \) vs. \( t \) with points plotted.
5. Table 1.3
Results of Euler's method

<table>
<thead>
<tr>
<th>k</th>
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<td>1</td>
<td>-1</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>-1</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>-1</td>
<td>0</td>
</tr>
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<td>-1</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>5</td>
<td>-1</td>
<td></td>
</tr>
</tbody>
</table>

7. Table 1.4
Results of Euler's method (shown rounded to two decimal places)

<table>
<thead>
<tr>
<th>k</th>
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<th>y_k</th>
<th>m_k</th>
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<tr>
<td>4</td>
<td>2.0</td>
<td>5.81</td>
<td></td>
</tr>
</tbody>
</table>

9. Because the differential equation is autonomous, the computation that determines y_{k+1} from y_k depends only on y_k and \Delta t and not on the actual value of t_k. Hence the approximate y-values that are obtained in both exercises are the same. It is useful to think about this fact in terms of the slope field of an autonomous equation.

11. As the solution approaches the equilibrium solution corresponding to w = 3, its slope decreases. We do not expect the solution to "jump over" an equilibrium solution (see the Existence and Uniqueness Theorem in Section 1.5).

13. Table 1.5
Results of Euler's method with \Delta t = 1.0
(shown to two decimal places)

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<td>2</td>
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</tr>
</tbody>
</table>
CHAPTER I FIRST-ORDER DIFFERENTIAL EQUATIONS

Table 1.6
Results of Euler's method with $\Delta t = 0.5$ (shown to two decimal places)

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<th>$m_k$</th>
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<th>$t_k$</th>
<th>$y_k$</th>
<th>$m_k$</th>
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</thead>
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<td>0.02</td>
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<td>0.5</td>
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<td>0.02</td>
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Table 1.7
Results of Euler's method with $\Delta t = 0.25$ (shown to two decimal places)

<table>
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From the differential equation, we see that $dy/dt$ is positive and decreasing as long as $y(0) = 1$ and $y(t) < 2$ for $t > 0$. Therefore, $y(t)$ is increasing, and its graph is concave down. Since Euler's method uses line segments to approximate the graph of the actual solution, the approximate solutions will always be greater than the actual solution. This error decreases as the step size decreases.
15. Graph of approximate solution obtained using Euler's method with $\Delta t = 0.1$.

17. Graph of approximate solution obtained using Euler's method with $\Delta t = 0.1$.

19. (a) (b)

- The roots of $p(y)$ correspond to the equilibrium solutions.
- From the graphs of the solutions, we see that there are three equilibrium points, hence three real roots. We can find two of the roots by using Euler's method with a positive value of $\Delta t$. The other root can be found using Euler's method with a negative $\Delta t$. Since solutions tend toward the root, we can use a fairly large step size to obtain the required accuracy, but since this is only an approximate solution, it is best to double check the location of the root by substituting the value into $p(y)$. The approximate roots are $y \approx 2.115$, $y \approx -1.861$ and $y \approx -0.254$.

**EXERCISES FOR SECTION 1.5**

1. Since the constant function $y_1(t) = 3$ for all $t$ is a solution, then the graph of any other solution $y(t)$ with $y(0) < 3$ cannot cross the line $y = 3$ by the Uniqueness Theorem. So $y(t) < 3$ for all $t$ in the domain of $y(t)$.

3. Because $y_2(0) < y(0) < y_1(0)$, we know that

$$-t^2 = y_2(t) < y(t) < y_1(t) = t + 2$$
CHAPTER 1 FIRST-ORDER DIFFERENTIAL EQUATIONS

for all $t$. This restricts how large positive or negative $y(t)$ can be for a given value of $t$ (that is, between $-t^2$ and $t + 2$). As $t \to -\infty$, $y(t) \to -\infty$ between $-t^2$ and $t + 2$ (as $y(t) \to -\infty$ as $t \to -\infty$ at least linearly, but no faster than quadratically).

5. The Existence Theorem implies that a solution with this initial condition exists, at least for a small $t$-interval about $t = 0$. This differential equation has equilibrium solutions $y_1(t) = 0$, $y_2(t) = 2$, and $y_3(t) = 3$. Since $y(0) = 4$, the Uniqueness Theorem implies that $y(t) > 3$ for all $t$ in the domain of $y(t)$. Also, $dy/dt > 0$ for all $y > 3$, so the solution $y(t)$ is increasing for all $t$ in its domain.

7. Because $0 < y(0) < 2$ and $y_1(t) = 0$ and $y_2(t) = 2$ are equilibrium solutions of the differential equation, we know that $0 < y(t) < 2$ for all $t$ by the Uniqueness Theorem. Also, $dy/dt > 0$ for $0 < y < 2$, so $dy/dt$ is always positive for this solution. Hence, $y(t) \to 2$ as $t \to \infty$, and $y(t) \to 0$ as $t \to -\infty$.

9. (a) To check that $y_1(t) = t^2$ is a solution, we compute

$$\frac{dy_1}{dt} = 2t$$

and

$$-y_1^2 + y_1 + 2y_1t^2 + 2t - t^2 - t^4 = -(t^2)^2 + (t^2) + 2(t^2)t^2 + 2t - t^2 - t^4 = 2t.$$ To check that $y_2(t) = t^2 + 1$ is a solution, we compute

$$\frac{dy_2}{dt} = 2t$$

and

$$-y_2^2 + y_2 + 2y_2t^2 + 2t - t^2 - t^4 = -(t^2 + 1)^2 + (t^2 + 1) + 2(t^2 + 1)t^2 + 2t - t^2 - t^4 = 2t.$$ (b) The initial values of the two solutions are $y_1(0) = 0$ and $y_2(0) = 1$. Thus if $y(t)$ is a solution and $y_1(0) = 0 < y(0) < 1 = y_2(0)$, then we can apply the Uniqueness Theorem to obtain

$$y_1(t) = t^2 < y(t) < t^2 + 1 = y_2(t)$$

for all $t$. Note that since the differential equation satisfies the hypothesis of the Existence and Uniqueness Theorem over the entire $ty$-plane, we can continue to extend the solution as long as it does not escape to $\pm \infty$ in finite time. Since it is bounded above and below by solutions that exist for all time, $y(t)$ is defined for all time also.
EXERCISES FOR SECTION 1.6

1. The equilibrium points of $dy/dt = f(y)$ are the numbers $y$ where $f(y) = 0$. For
   
   \[ f(y) = 3y(1 - y), \]

   the equilibrium points are $y = 0$ and $y = 1$. Since $f(y)$ is negative for $y < 0$, positive for $0 < y < 1$, and negative for $y > 1$, the equilibrium point $y = 0$ is a source and the equilibrium point $y = 1$ is a sink.

   \[ y = 1 \quad \text{sink} \]
   \[ y = 0 \quad \text{source} \]

3. The equilibrium points of $dy/dt = f(y)$ are the numbers $y$ where $f(y) = 0$. For $f(y) = \cos y$, the equilibrium points are $y = \pi/2 + n\pi$, where $n = 0, \pm 1, \pm 2, \ldots$. Since $\cos y > 0$ for $-\pi/2 < y < \pi/2$ and $\cos y < 0$ for $\pi/2 < y < 3\pi/2$, we see that the equilibrium point at $y = \pi/2$ is a sink. Since the sign of $\cos y$ alternates between positive and negative in a period fashion, we see that the equilibrium points at $y = \pi/2 + 2n\pi$ are sinks and the equilibrium points at $y = 3\pi/2 + 2n\pi$ are sources.

   \[ y = 3\pi/2 \quad \text{source} \]
   \[ y = \pi/2 \quad \text{sink} \]
   \[ y = -\pi/2 \quad \text{source} \]

5. The equilibrium points of $dw/dt = f(w)$ are the numbers $w$ where $f(w) = 0$. For $f(w) = (w-2)\sin w$, the equilibrium points are $w = 2$ and $w = n\pi$, where $n = 0, \pm 1, \pm 2, \ldots$. The sign of $(w-2)\sin w$ alternates between positive and negative at successive zeros. It is positive for $-\pi < w < 0$ and negative for $0 < w < 2$. Therefore, $w = 0$ is a sink, and the equilibrium points alternate back and forth between sources and sinks.

   \[ w = \pi \quad \text{sink} \]
   \[ w = 2 \quad \text{source} \]
   \[ w = 0 \quad \text{sink} \]

7. The derivative $dw/dt$ is always positive, so there are no equilibrium points, and all solutions increase for all time.
9. The equilibrium points of \( \frac{dy}{dt} = f(y) \) are the numbers \( y \) where \( f(y) = 0 \). For \( f(y) = -1 + \cos y \), the equilibrium points are \( y = 2n\pi \), where \( n = 0, \pm 1, \pm 2, \ldots \). Since \( f(y) \) is nonpositive for all values of \( y \), all of the equilibrium points are nodes.

\[
\begin{align*}
y &= 2\pi & \text{node} \\
y &= 0 & \text{node} \\
y &= -2\pi & \text{node}
\end{align*}
\]

11. The equilibrium points of \( \frac{dy}{dt} = f(y) \) are the numbers \( y \) where \( f(y) = 0 \). For \( f(y) = y \ln |y| \), there are equilibrium points at \( y = \pm 1 \). In addition, although the function \( f(y) \) is technically undefined at \( y = 0 \), the limit of \( f(y) \) as \( y \to 0 \) is 0. Thus we can treat \( y = 0 \) as another equilibrium point. Since \( f(y) < 0 \) for \( y < -1 \) and \( 0 < y < 1 \), and \( f(y) > 0 \) for \( y > 1 \) and \( -1 < y < 0 \), \( y = -1 \) is a source, \( y = 0 \) is a sink, and \( y = 1 \) is a source.

\[
\begin{align*}
y &= 1 & \text{source} \\
y &= 0 & \text{sink} \\
y &= -1 & \text{source}
\end{align*}
\]

13. \[
\begin{array}{c}
\text{Graph 13.}
\end{array}
\]

15. \[
\begin{array}{c}
\text{Graph 15.}
\end{array}
\]

17. \[
\begin{array}{c}
\text{Graph 17.}
\end{array}
\]

19. \[
\begin{array}{c}
\text{Graph 19.}
\end{array}
\]
The initial value $y(0) = 1$ is between the equilibrium points $y = 2 - \sqrt{2}$ and $y = 2 + \sqrt{2}$. Also, $dy/dt < 0$ for $2 - \sqrt{2} < y < 2 + \sqrt{2}$. Hence the solution is decreasing and tends toward the smaller equilibrium point $y = 2 - \sqrt{2}$ as $t \to \infty$. It tends toward the larger equilibrium point $y = 2 + \sqrt{2}$ as $t \to -\infty$.

The initial value $y(0) = -10$ is below both of the equilibrium points. Since $dy/dt$ is positive for $y < 2 - \sqrt{2}$, the solution is increasing for all $t$ and tends to the equilibrium point $y = 2 - \sqrt{2}$ as $t \to \infty$. As $t$ decreases, it becomes unbounded in the negative direction in finite time.

The initial value $y(3) = 1$ is between the equilibrium points $y = 2 - \sqrt{2}$ and $y = 2 + \sqrt{2}$. Also, $dy/dt < 0$ for $2 - \sqrt{2} < y < 2 + \sqrt{2}$. Hence the solution is decreasing and tends toward the smaller equilibrium point $y = 2 - \sqrt{2}$ as $t \to \infty$. It tends toward the larger equilibrium point $y = 2 + \sqrt{2}$ as $t \to -\infty$.

The function $f(y)$ has two zeros $\pm y_0$, where $y_0$ is some positive number. So the differential equation $dy/dt = f(y)$ has two equilibrium solutions, one for each zero. Also, $f(y) < 0$ if $-y_0 < y < y_0$ and $f(y) > 0$ if $y < -y_0$ or if $y > y_0$. Hence $y_0$ is a source and $-y_0$ is a sink.

The function $f(y)$ has two zeros, one positive and one negative. We denote them as $y_1$ and $y_2$, where $y_1 < y_2$. So the differential equation $dy/dt = f(y)$ has two equilibrium solutions, one for each zero. Also, $f(y) > 0$ if $y_1 < y < y_2$ and $f(y) < 0$ if $y < y_1$ or if $y > y_2$. Hence $y_1$ is a source and $y_2$ is a sink.
33. Since there are two equilibrium points, the graph of \( f(y) \) must touch the y-axis at two distinct numbers \( y_1 \) and \( y_2 \). Assume that \( y_1 < y_2 \). Since the arrows point up if \( y < y_1 \) and if \( y > y_2 \), we must have \( f(y) > 0 \) for \( y < y_1 \) and for \( y > y_2 \). Similarly, \( f(y) < 0 \) for \( y_1 < y < y_2 \).

The precise location of the equilibrium points is not given, and the direction of the arrows on the phase line is determined only by the sign (and not the magnitude) of \( f(y) \). So the following graph is one of many possible answers.

\[
\begin{align*}
  f(y) \\
  \hline
  y
\end{align*}
\]

35. Since there are four equilibrium points, the graph of \( f(y) \) must touch the y-axis at four distinct numbers \( y_1, y_2, y_3, \) and \( y_4 \). We assume that \( y_1 < y_2 < y_3 < y_4 \). Since the arrows point up only if \( y_1 < y < y_2 \) or if \( y_2 < y < y_3 \), we must have \( f(y) > 0 \) for \( y_1 < y < y_2 \) and for \( y_2 < y < y_3 \). Moreover, \( f(y) < 0 \) if \( y < y_1 \), if \( y_3 < y < y_4 \), or if \( y > y_4 \). Therefore, the graph of \( f \) crosses the y-axis at \( y_1 \) and \( y_3 \), but it is tangent to the y-axis at \( y_2 \) and \( y_4 \).

The precise location of the equilibrium points is not given, and the direction of the arrows on the phase line is determined only by the sign (and not the magnitude) of \( f(y) \). So the following graph is one of many possible answers.

\[
\begin{align*}
  f(y) \\
  \hline
  y
\end{align*}
\]

37. (a) In terms of the phase line with \( P \geq 0 \), there are three equilibrium points. If we assume that \( f(P) \) is differentiable, then a decreasing population at \( P = 100 \) implies that \( f(P) < 0 \) for \( P > 50 \). An increasing population at \( P = 25 \) implies that \( f(P) > 0 \) for \( 10 < P < 50 \). These assumptions leave two possible phase lines since the arrow between \( P = 0 \) and \( P = 10 \) is undetermined.

\[
\begin{align*}
  & P = 50 \\
  & P = 10 \\
  & P = 0
\end{align*}
\]
(b) Given the observations in part (a), we see that there are two basic types of graphs that go with the assumptions. However, there are many graphs that correspond to each possibility. The following two graphs are representative.

(c) The functions \( f(P) = P(P - 10)(50 - P) \) and \( f(P) = P(P - 10)^2(50 - P) \) respectively are two examples but there are many others.

39. The equilibrium points occur at solutions of \( dy/dt = y^2 + a = 0 \). For \( a > 0 \), there are no equilibrium points. For \( a = 0 \), there is one equilibrium point, \( y = 0 \). For \( a < 0 \), there are two equilibrium points, \( y = \pm \sqrt{-a} \).

To draw the phase lines, note that:

- If \( a > 0 \), \( dy/dt = y^2 + a > 0 \), so the solutions are always increasing.
- If \( a = 0 \), \( dy/dt > 0 \) unless \( y = 0 \). Thus, \( y = 0 \) is a node.
- For \( a < 0 \), \( dy/dt < 0 \) for \( -\sqrt{-a} < y < \sqrt{-a} \), and \( dy/dt > 0 \) for \( y < -\sqrt{-a} \) and for \( y > \sqrt{-a} \).

(a) The phase lines for \( a < 0 \) are qualitatively the same, and the phase lines for \( a > 0 \) are qualitatively the same.

(b) The phase line undergoes a qualitative change at \( a = 0 \).

41. (a) Because \( f(y) \) is continuous we can use the Intermediate Value Theorem to say that there must be a zero of \( f(y) \) between \(-10\) and \(10\). This value of \( y \) is an equilibrium point of the differential equation. In fact, \( f(y) \) must cross from positive to negative, so if there is a single equilibrium point, it must be a sink (see part (b)).

(b) We know that \( f(y) \) must cross the \( y \)-axis between \(-10\) and \(10\). Moreover, it must cross from positive to negative because \( f(-10) \) is positive and \( f(10) \) is negative. Where \( f(y) \) crosses the
y-axis from positive to negative, we have a sink. If \( y = 1 \) is a source, then crosses the y-axis from negative to positive at \( y = 1 \). Hence, \( f(y) \) must cross the y-axis from positive to negative at least once between \( y = -10 \) and \( y = 1 \) and at least once between \( y = 1 \) and \( y = 10 \). There

must be at least two sinks in each of these intervals. (We need the assumption that the number of equilibrium points is finite to prevent cases where \( f(y) = 0 \) along an entire interval.)

43. (a) Because the first and second derivative are zero at \( y_0 \) and the third derivative is positive, Taylor’s Theorem implies that the function \( f(y) \) is approximately equal to

\[
\frac{f'''(y_0)}{3!}(y - y_0)^3
\]

for \( y \) near \( y_0 \). Since \( f'''(y_0) > 0 \), \( f(y) \) is increasing near \( y_0 \). Hence, \( y_0 \) is a source.

(b) Just as in part (a), we see that \( f(y) \) is decreasing near \( y_0 \), so \( y_0 \) is a sink.

(c) In this case, we can approximate \( f(y) \) near \( y_0 \) by

\[
\frac{f''(y_0)}{2!}(y - y_0)^2.
\]

Since the second derivative of \( f(y) \) at \( y_0 \) is assumed to be positive, \( f(y) \) is positive on both sides of \( y_0 \) for \( y \) near \( y_0 \). Hence \( y_0 \) is a node.

45. One assumption of the model is that, if no people are present, then the time between trains decreases at a constant rate. Hence the term \(-\alpha\) represents this assumption. The parameter \( \alpha \) should be positive, so that \(-\alpha\) makes a negative contribution to \( dx/dt \).

The term \( \beta x \) represents the effect of the passengers. The parameter \( \beta \) should be positive so that \( \beta x \) contributes positively to \( dx/dt \).

47. Note that the only equilibrium point is a source. If the initial gap between trains is too large, then \( x \) will increase without bound. If it is too small, \( x \) will decrease to zero. When \( x = 0 \), the two trains are next to each other, and they will stay together since \( x < 0 \) is not physically possible in this problem.

If the time between trains is exactly the equilibrium value (\( x = \alpha/\beta \)), then theoretically \( x(t) \) is constant. However, any disruption to \( x \) causes the solution to tend away from the source. Since it is very likely that some stops will have fewer than the expected number of passengers and some stops will have more, it is unlikely that the time between trains will remain constant for long.

**EXERCISES FOR SECTION 1.7**

1. The equilibrium points occur at solutions of \( dy/dt = y^2 + a = 0 \). For \( a > 0 \), there are no equilibrium points. For \( a = 0 \), there is one equilibrium point, \( y = 0 \). For \( a < 0 \), there are two equilibrium points, \( y = \pm \sqrt{-a} \). Thus, \( a = 0 \) is a bifurcation value.

To draw the phase lines, note that:

- If \( a > 0 \), \( dy/dt = y^2 + a > 0 \), so the solutions are always increasing.
- If \( a = 0 \), \( dy/dt > 0 \) unless \( y = 0 \). Thus, \( y = 0 \) is a node.