SOME RESULTS ON THE REPRESENTATION THEORY OF VERTEX OPERATOR ALGEBRAS AND INTEGER PARTITION IDENTITIES

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INTRODUCTION

PARTITIONS

$$
\begin{aligned}
9 &= 9 \\
&= 8 + 1 \\
&= 7 + 2 \\
&= 6 + 3 \\
&= 5 + 3 + 1
\end{aligned}
$$

There are total 30 such ways.

RR 1

Partitions of n such that each part $\equiv 1, 4 \pmod 5$
are equinumerous with
partitions whose adjacent parts differ by at least 2.

RR 2

Partitions of n such that each part $\equiv 2, 3 \pmod 5$
are equinumerous with
partitions whose adjacent parts differ by at least 2 and
whose smallest part is at least 2.

Rogers-Ramanujan 1

$$\begin{aligned} 9 &= 9 \\ &= 8+1 \\ &= 7+2 \\ &= 6+3 \\ &= 5+3+1 \end{aligned}$$

$$\begin{aligned} 9 &= 9 \\ &= 6+1+1+1 \\ &= 4+4+1 \\ &= 4+1+1+1+1+1 \\ &= 1+1+1+1+1+1+1+1+1 \end{aligned}$$

Rogers-Ramanujan 2

$$\begin{aligned} 9 &= 9 \\ &= 7+2 \\ &= 6+3 \end{aligned}$$

$$\begin{aligned} 9 &= 7+2 \\ &= 3+3+3 \\ &= 3+2+2+2 \end{aligned}$$

$d_i(n)$

Number of partitions of n with adjacent parts differing by at least 2, such that the smallest allowed part is i.

RR 1

$$\sum_{n\geq 0} d_1(n)q^n = \frac{1}{(1-q)(1-q^4)(1-q^6)(1-q^9)\cdots}$$

RR 2

$$\sum_{n\geq 0} d_2(n)q^n = \frac{1}{(1-q^2)(1-q^3)(1-q^7)(1-q^8)\cdots}$$

GÖ-GO 2

Partitions of n with each part $\equiv 1, 4, 7 \pmod 8$
are equinumerous with
partitions whose adjacent parts differ by at least 2 with
difference at least 4 for adjacent *even* parts.

$20 = 9 + 6 + 4 + 1 \quad \times$

GÖ-GO 2

Partitions of n with each part $\equiv 3, 4, 5 \pmod 8$
are equinumerous with
partitions whose adjacent parts differ by at least 2 with
difference at least 4 for adjacent *even* parts,
moreover, smallest allowed part is 3.

Lepowsky-Milne '78

Consider $A_1^{(1)}$, principal specialization.

$$\begin{aligned}\mathrm{ch}(3\Lambda_0) = \mathrm{ch}(3\Lambda_1) \\ &= F(q) \cdot \frac{1}{(1-q)(1-q^4)(1-q^6)(1-q^9)\cdots} \\ \mathrm{ch}(2\Lambda_0 + \Lambda_1) = \mathrm{ch}(\Lambda_0 + 2\Lambda_1) \\ &= F(q) \cdot \frac{1}{(1-q^2)(1-q^3)(1-q^7)(1-q^8)\cdots}\end{aligned}$$

Lepowsky-Wilson '78-'85

Explained the RR sum sides and proved the RR identities by inventing **principal Heisenberg subalgebras** and **Z-algebras**.

Identities	Algebra	Level
Rogers-Ramanujan	$A_1^{(1)}$	3
Gordon-Andrews	$A_1^{(1)}$	Odd
Andrews-Bressoud	$A_1^{(1)}$	Even
Capparelli	$A_2^{(2)}$	3
Nandi's conjectures	$A_2^{(2)}$	4
Göllnitz-Gordon	$A_5^{(2)}$	2
K.-Russell Conjectures	$D_4^{(3)}$	3

Feigin-Stoyanovsky '93-'94

A certain subspace $W_\Lambda \subset L(\Lambda)$ exhibits difference 2 conditions.

RR sum sides for level 1, Gordon-Andrews sum sides at higher levels.

Products came from geometric arguments.

Capparelli-Lepowsky-Milas '01

Explained the recursion $F(x,q) = F(xq,q) + xqF(xq^2,q)$ using exact sequences among $W(\Lambda_0)$ and $W(\Lambda_1)$.

Intertwining operators gave the maps.

Calinescu-Lepowsky-Milas '08

Proved the assumed presentation result for W_{Λ_i}.

DIRECTIONS

- Finding new identities: **K.-Russell**
- Finding "motivated proofs:" **K.-Lepowsky-Russell-Sills**, **Coulson-K.-Lepowsky-McRae-Qi-Russell-Sadowski**
- Interpreting these proofs using the theory of vertex algebras: **Thesis**
- In vertex operator algebra theory — PRODUCTS TO SUMS: **Thesis**
- In vertex operator algebra theory — SUMS TO PRODUCTS: **Thesis**

SUMS TO PRODUCTS

$$\mathcal{A} = \mathbb{C}[x_{-1}, x_{-2}, \dots]$$

$$I_{\Lambda_0} = \text{Ideal generated by } \left\{ r_{-n} = \sum_{i=1}^{n-1} x_{-i} x_{-n+i} \,;\, n \geq 2 \right\}.$$

$$\begin{aligned}
r_{-2} &= x_{-1}x_{-1} &&= x_{-1}^2 \\
r_{-3} &= x_{-1}x_{-2} + x_{-2}x_{-1} &&= 2x_{-1}x_{-2} \\
r_{-4} &= x_{-1}x_{-3} + x_{-2}x_{-3} + x_{-3}x_{-1} &&= 2x_{-1}x_{-3} + x_{-2}^2
\end{aligned}$$

Definition: Principal Subspace

We call $W_{\Lambda_0} = \mathcal{A}/I_{\Lambda_0}$ a *principal subspace*.

Recall:

$$\begin{aligned} r_{-2} &= x_{-1}x_{-1} &&= x_{-1}^2 \\ r_{-3} &= x_{-1}x_{-2} + x_{-2}x_{-1} &&= 2x_{-1}x_{-2} \\ r_{-4} &= x_{-1}x_{-3} + x_{-2}x_{-3} + x_{-3}x_{-1} &&= 2x_{-1}x_{-3} + x_{-2}^2 \end{aligned}$$

and so on ....

- Has a *basis* of monomials satisfying difference-2 conditions. For a proof using Gröbner bases, see Bruschek-Mourtada-Schepers '13. Slightly different space considered.
- In this paper, it comes up while calculating Hilbert-Poincaré series of arc space of a double point.
- Shows up in a lot of different problems — we'll see that later.

Question

Where are the products?

Idea (J. Lepowsky)

First use the Jacobi triple product identity

$$\frac{1}{(1-q)(1-q^4)(1-q^6)(1-q^9)\cdots} = \sum_{\lambda \geq 0} (-1)^\lambda \cdot \frac{q^{\lambda(5\lambda-1)/2}(1+q^\lambda)}{\prod_{n\geq 1}(1-q^n)}$$

Observe that this is an alternating sum:

Could be explained via Euler-Poincaré principle applied to a resolution.

KOSZUL COMPLEX

$$\cdots C_3 = \bigoplus_{i_1, i_2 \leq -2} \mathcal{A}\xi_{i_1, i_2} \xrightarrow{\partial_3} C_2 = \bigoplus_{i_1 \leq -2} \mathcal{A}\xi_{i_1} \xrightarrow{\partial_2} C_1 = \mathcal{A} \xrightarrow{\partial_1} C_0 = W_{\Lambda_0} \twoheadrightarrow 0$$

- $\xi_{\cdots, i, \cdots, j, \cdots} = -\xi_{\cdots, j, \cdots, i, \cdots}$.
- ∂_1 is the projection map $\mathcal{A} \longrightarrow \mathcal{A}/I_{\Lambda_0}$.
- $\partial_{k+1}(\xi_{-i_1, -i_2, \cdots, -i_k}) = \sum_{n=1}^{k} (-1)^{n-1} \cdot r_{-i_n} \cdot \xi_{-i_1, -i_2, \cdots, \widehat{-i_n}, \cdots, -i_k}$.
  $\partial(r_{-i}) = 0$
  $\partial(\xi_{-i}) = r_{-i}$
  $\partial(\xi_{-i,-j}) = r_{-i}\xi_{-j} - r_{-j}\xi_{-i}$
- $H_n = \mathrm{Ker}(\partial_n)/\mathrm{Im}(\partial_{n+1})$
  $H_0 = H_1 = 0$.

$$\cdots C_3 = \bigoplus_{i_1, i_2 \leq -2} \mathcal{A}\xi_{i_1, i_2} \xrightarrow{\partial_3} C_2 = \bigoplus_{i_1 \leq -2} \mathcal{A}\xi_{i_1} \xrightarrow{\partial_2} C_1 = \mathcal{A} \xrightarrow{\partial_1} C_0 = W_{\Lambda_0} \twoheadrightarrow 0$$

Interpretation

$$\partial_2 : \bigoplus_{i_1 \leq -2} \mathcal{A}\xi_{i_1} \longrightarrow \mathcal{A}$$

$\mathrm{Ker}(\partial_2)$ is precisely the *the space of relations* amongst the r_ns.

$$\partial_3 : \bigoplus_{i_1, i_2 \leq -2} \mathcal{A}\xi_{i_1 i_2} \longrightarrow \bigoplus_{i_1 \leq -2} \mathcal{A}\xi_{i_1}$$

$\mathrm{Im}(\partial_3)$ is precisely the the space of *trivial* relations amongst the r_ns: $r_n \cdot r_m - r_m \cdot r_n = 0$.

$H_2 = \mathrm{Ker}(\partial_2)/\mathrm{Im}(\partial_3)$ measures the space of "non-trivial" relations.

$$\cdots C_3 = \bigoplus_{i_1, i_2 \leq -2} \mathcal{A}\xi_{i_1, i_2} \xrightarrow{\partial_3} C_2 = \bigoplus_{i_1 \leq -2} \mathcal{A}\xi_{i_1} \xrightarrow{\partial_2} C_1 = \mathcal{A} \xrightarrow{\partial_1} C_0 = W_{\Lambda_0} \twoheadrightarrow 0$$

Euler-Poincaré principle

With χ being the "dimension" (actually, the (x, q)–character)

$$\chi(W_{\Lambda_0}; x, q) = \sum_{n \geq 1} (-1)^{n+1} \left(\chi(C_n; x, q) - \chi(H_n; x, q)\right).$$

The Problem

Find the precise structure of H_ns and calculate $\chi(H_n; x, q)$.

There is a derivation L_{-1} of $\mathcal{A}$, that can be extended to the C_j's:

$$\begin{aligned}
L_{-1} \cdot x_{-j} &= j x_{-j-1} \\
L_{-1} \cdot r_{-j} &= (j-1) r_{-j-1} \\
L_{-1} \cdot (\xi_{-i_1,\cdots,-i_k}) &= (i_1 - 1)\xi_{-i_1-1,-i_2,\cdots,-i_k} + (i_2 - 1)\xi_{-i_1,-i_2-1,\cdots,-i_k} \\
&\quad + \cdots + (i_k - 1)\xi_{-i_1,-i_2\cdots,-i_k-1}
\end{aligned}$$

$$L_{-1}(a \cdot c) = L_{-1}(a)c + aL_{-1}(c)$$

$$\begin{aligned}
[\partial, L_{-1}] &= 0 \\
L_{-1}\mathrm{Ker}(\partial) &\subset \mathrm{Ker}(\partial) \\
L_{-1}\mathrm{Im}(\partial) &\subset \mathrm{Im}(\partial)
\end{aligned}$$

There is an automorphism σ of $\mathcal{A}$, that can be extended to the C_js:

$$\sigma(x_{-1}) = 0,$$
$$\sigma(x_{-i}) = x_{-i+1}$$
$$\sigma(\xi_{-i_1,\cdots,-i_k}) = \xi_{-i_1+2,\cdots,-i_k+2}$$

$$\sigma(a \cdot c) = \sigma(a)\sigma(c)$$

$$[\partial, \sigma] = 0$$
$$\sigma \mathrm{Ker}(\partial) \subset \mathrm{Ker}(\partial)$$
$$\sigma \mathrm{Im}(\partial) \subset \mathrm{Im}(\partial)$$

Some obvious elements in the $\mathrm{Ker}(\partial_2)$:

$$\mu_{-4} = 2x_{-2}\xi_{-2} - x_{-1}\xi_{-3}$$
$$\partial(\mu_{-4}) = 2x_{-2} \cdot x_{-1}^2 - x_{-1} \cdot 2x_{-1}x_{-2} = 0$$

$$\partial(L_{-1}^s \cdot \mu_{-4}) = 0 \quad \text{for } s \in \mathbb{N}.$$

$$\begin{aligned}
\mu_{-4} &= 2x_{-2}\xi_{-2} - x_{-1}\xi_{-3} \\
\mu_{-5} &= 4x_{-3}\xi_{-2} + x_{-2}\xi_{-3} - 2x_{-1}\xi_{-4} \\
\mu_{-6} &= 6x_{-4}\xi_{-2} + 3x_{-3}\xi_{-3} - 3x_{-1}\xi_{-5} \\
\mu_{-7} &= 8x_{-5}\xi_{-2} + 5x_{-4}\xi_{-3} + 2x_{-3}\xi_{-4} - 1x_{-2}\xi_{-5} - 4x_{-1}\xi_{-6} \\
\mu_{-8} &= 10x_{-6}\xi_{-2} + 7x_{-5}\xi_{-3} + 4x_{-4}\xi_{-4} + 1x_{-3}\xi_{-5} - 2x_{-2}\xi_{-6} - 5x_{-1}\xi_{-7}
\end{aligned}$$

and so on...

Theorem (K.)

The second homology H_2 is generated by the elements $L_{-1}^s \cdot \mu_{-4}$ for $s \in \mathbb{N}$.

Remark

The proof is very similar to the proof of presentation of W_{Λ_0} in Calinescu-Lepowsky-Milas '08.

Use the "minimal counter-example" technique.

Remark

The derivation L_{-1} does not enter the proof. Only the automorphism σ is needed.

Recall:

$$\sigma: \quad x_{-1} \mapsto 0, \quad x_{-j} \mapsto x_{-j+1}, \quad \xi_{-2} \mapsto 0, \quad \xi_{-3} \mapsto 0, \quad \xi_{-i} \mapsto \xi_{-i+2}.$$

Reason:

$\mu_{-4}, \mu_{-5}, \mu_{-6}$ determine the elements of $\mathrm{Ker}(\partial_2)$ of charge 3.

ORIGINS - PRINCIPAL SUBSPACES

$$\begin{aligned}
\mathfrak{g} &= \mathfrak{sl}_2 = \mathbb{C}\{x_\alpha, \alpha, x_{-\alpha}\} \\
\langle a, b\rangle &= \mathrm{Tr}(ab) \\
\mathfrak{n} &= \mathbb{C}\, x_\alpha
\end{aligned}$$

$$\begin{aligned}
\widehat{\mathfrak{g}} &= \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c \\
[x \otimes t^n, y \otimes t^m] &= [x, y] \otimes t^{m+n} + \langle x, y\rangle n\delta_{m+n,0}c \\
[c, \widehat{\mathfrak{g}}] &= 0 \\
\widehat{\mathfrak{g}} &\cong A_1^{(1)}
\end{aligned}$$

$$\begin{aligned}
\bar{\mathfrak{n}} &= \mathfrak{n} \otimes \mathbb{C}[t, t^{-1}] \subset \widehat{\mathfrak{g}} \\
\bar{\mathfrak{n}}_- &= \mathfrak{n} \otimes t^{-1}\mathbb{C}[t^{-1}] \subset \widehat{\mathfrak{g}} \\
[\bar{\mathfrak{n}}_-, \bar{\mathfrak{n}}_-] &= 0.
\end{aligned}$$

$$L(\Lambda): \text{ Irreducible, integrable } \widehat{\mathfrak{g}} - \text{module generated by highest wt. vector } v_\Lambda$$

$$W_\Lambda := \mathcal{U}(\bar{\mathfrak{n}}) \cdot v_\Lambda \cong \mathcal{U}(\bar{\mathfrak{n}}_-) \cdot v_\Lambda$$

$$\mathcal{U}(\bar{\mathfrak{n}}_-) \cong \mathbb{C}[x_{-1}, x_{-2}, \dots] = \mathcal{A}$$

$$f_\Lambda : \mathcal{U}(\bar{\mathfrak{n}}_-) \longrightarrow W_\Lambda$$
$$1 \mapsto v_\Lambda$$

Theorem (Calinescu-Lepowsky-Milas '08)

$\mathrm{Ker}(f_{\Lambda_0}) = I_{\Lambda_0}$ and $\mathrm{Ker}(f_{\Lambda_1}) = I_{\Lambda_0} + \mathcal{A}x_{-1}$.

RELATIONS TO GARLAND-LEPOWSKY RESOLUTION

$$\cdots \longrightarrow E_2 \longrightarrow E_1 \longrightarrow E_0 \longrightarrow L_{\Lambda_0}$$

$$v_{r_0 r_1 \cdot \Lambda_0} \qquad v_{r_0 \cdot \Lambda_0} \qquad v_{\Lambda_0} \qquad v_{\Lambda_0}$$

$$v_{r_0 r_1 \cdot \Lambda_0} \longmapsto (2x_{-2} - x_{-1}L(-1))v_{r_0 \cdot \Lambda_0}$$

$$v_{r_0 \Lambda_0} \longmapsto x_{-1}^2 v_{\Lambda_0}$$

$$w \cdot \Lambda = w(\Lambda + \rho) - \rho$$

RELATIONS TO KHOVANOV HOMOLOGY OF TORUS KNOTS

Stable unreduced Khovanov homology.

Apparently quite hard to compute for Torus knots. See `katlas.org`

$\mathrm{Kh}(T(n,\infty)) = \lim_{m\to\infty} q^{-(n-1)(m-1)+1}\mathrm{Kh}(T(n,m))$.

This limit exists (Stošić).

Conjecture (Gorsky-Oblomkov-Rasmussen '12)

$\mathrm{Kh}(T(n,\infty))$ is dual to the homology of the Koszul complex determined by the elements $r_{-2},\ldots,r_{-n-1}$.

Every Koszul complex is a dg-algebra:

$\xi_{-i_1,\ldots,-i_j} = \xi_{-i_1} \wedge \cdots \wedge \xi_{-i_j}$.

$Z = \bigoplus_{n \geq 0} \mathrm{Ker}(\partial_n)$ is a sub-algebra with

$B = \bigoplus_{n \geq 0} \mathrm{Im}(\partial_n)$, a two-sided ideal.

$H = \bigoplus_{n \geq 0} H_n$ is a graded algebra.

Conjecture (Gorsky-Oblomkov-Rasmussen '12)

For $T(n,\infty)$, H is generated as by the elements $\mu_{-4},\dots,\mu_{-n-2}$, with the defining relations being

$$\begin{aligned} x(z)^2 &= 0 \\ x(z)\mu(z) &= 0 \\ x''(z)\mu(z) - x'(z)\mu'(z) &= 0 \\ \mu(z)\mu'(z) &= 0, \end{aligned}$$

recall that $\mu(z) = 2x'(z)\xi(z) - x(z)\xi'(z)$.

σ: unreduced $\longrightarrow$ reduced.

$$
\begin{aligned}
E_0 &\longrightarrow L(\Lambda_0) \\
R &= \mathcal{U}(\mathfrak{g})x^2_{-1}\mathbf{1} \subset E_0 \\
\bar{R} &= \text{ Coeffs of } r(x) = Y(r,x), r \in R \\
\bar{R}\mathbf{1} \otimes E_0 &\longrightarrow E_0 \\
u \otimes v &\longmapsto u_{-1}v \\
Y(u,x) \otimes Y(v,x) &\longmapsto \vdots Y(u,x)Y(v,x) \vdots, u \in \bar{R}\mathbf{1}
\end{aligned}
$$

Meurman-Primc '99, Primc '02

The kernel of the map above is generated, in some sense, by:

$$
\begin{aligned}
&\frac{d}{dx}\left(x_\theta(x)^2\right) \otimes x_\theta(x) - 2(x_\theta(z)^2) \otimes \frac{d}{dx}x_\theta(z) \\
&L(-1)x^2_{-1}\mathbf{1} \otimes x_{-1}\mathbf{1} - 2x^2_{-1}\mathbf{1} \otimes L(-1)x_{-1}\mathbf{1}
\end{aligned}
$$

PRODUCTS TO SUMS

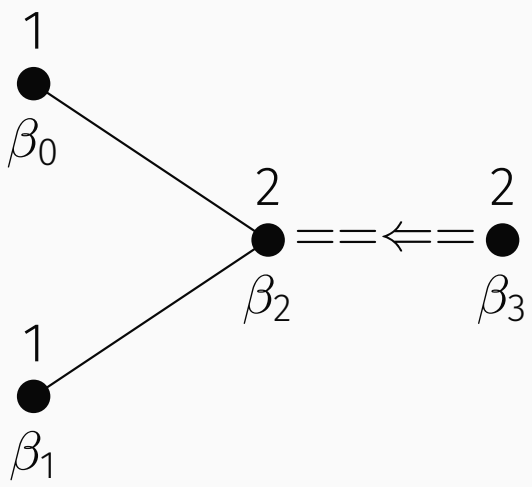

Figure 1: $A_5^{(2)}$

$$\mathrm{ch}(\Omega(L(\Lambda_0+\Lambda_1))) = \frac{1}{(1-q)(1-q^4)(1-q^7)\cdots}$$

$$\mathrm{ch}(\Omega(L(\Lambda_3))) = \frac{1}{(1-q^3)(1-q^4)(1-q^5)\cdots}$$

Theorem (K.)

Spanning works!

GÖLLNITZ-GORDON IDENTITIES AND $A_5^{(2)}$

- Higher rank motivated proofs?
- Multi-colour generalizations of the Göllnitz-Gordon identities?
- Look at level 2 standard modules contained in the tensor product of two inequivalent level 1 standard modules for higher rank algebras in the same "series."

$A_5^{(2)}$	Göllnitz-Gordon
$A_7^{(2)}$	Rogers-Ramanujan [Bos-Misra '94]
$A_9^{(2)}$	?????
$A_{11}^{(2)}$	Nandi's identities

PRODUCTS TO SUMS, AGAIN

RR 1

Partitions of n such that each part $\equiv 1, 4 \pmod 5$
are equinumerous with
partitions whose adjacent parts differ by at least 2.

RR 2

Partitions of n such that each part $\equiv 2, 3 \pmod 5$
are equinumerous with
partitions whose adjacent parts differ by at least 2 and
whose smallest part is at least 2.

Question: Ehrenpreis

RR1 $\geq$ RR2: Reason — sum sides.

Can one see this using *only* products?

Motivated by Ehrenpreis's question:

(RR1 $\geq$ RR2, *without* looking at sum-sides)

$$G_1 = \prod_{\substack{m>0,\\ m \not\equiv 0,\pm 2(5)}} \frac{1}{(1-q^m)} = 1 + q + q^2 + q^3 + 2q^4 + 2q^5 + 3q^6 + 3q^7 + \cdots$$

$$G_2 = \prod_{\substack{m>0,\\ m \not\equiv 0,\pm 1(5)}} \frac{1}{(1-q^m)} = 1 + \quad + q^2 + q^3 + \ q^4 + \ q^5 + 2q^6 + 2q^7 + \cdots$$

Observe: $G_1 \geq G_2$.

$G_1 - G_2 = q + q^4 + q^5 + q^6 + q^7 + 2q^8 + 2q^9 + \cdots$.

Let $G_3 = (G_1 - G_2)/q = 1 + q^3 + q^4 + q^5 + q^6 + 2q^7 + 2q^8 + \cdots$.

Let $\mathrm{G}_4 = (\mathrm{G}_2 - \mathrm{G}_3)/\mathrm{q}^2 = 1 + \mathrm{q}^4 + \mathrm{q}^5 + \mathrm{q}^6 + 2\mathrm{q}^7 + 2\mathrm{q}^8 + \cdots$.

Continue: Let $G_i = (G_{i-2} - G_{i-1})/q^{i-2}$.

Observe that each

$$G_i = 1 + q^i + \cdots \qquad \textbf{Empirical Hypothesis}$$

Proof of the Empirical Hypothesis not only answers Ehrenpreis's question but also immediately leads to a proof of RR.

Idea (J. Lepowsky, A. Milas)

Explain the sequence of G_i's using exact sequences similarly to the principal subspace story:

$$0 \longrightarrow \Omega(L_{G_2}) \longrightarrow \Omega(L_{G_1}) \longrightarrow ? \longrightarrow 0.$$

Use *twisted relativized intertwining operators*.

Theorem (K.)

There exist *explicitly constructed* twisted intertwining operators among certain mixed triples of untwisted and θ-twisted modules for the vertex operator algebra $V = L_{\widehat{\mathfrak{sl}_2}}(\ell, 0)$, for a positive integer ℓ.

Theorem (K.)

$V = V_{\mathbb{Z}\alpha}$: Rank 1 lattice vertex algebra, $\langle \alpha, \alpha \rangle = 2$.

θ: Lift to V of the "-1" automorphism of $\mathbb{Z}\alpha$.

$G = \mathbb{Z}/8$.

There exists a natural abelian intertwining algebra structure on the space $V \oplus V_{(\mathbb{Z}+1/2)\alpha} \oplus V^{T_1} \oplus V^{T_2}$ with grading group G.

The Y map for the abelian intertwining algebra is comprised of twisted intertwining operators.

Questions?