Problems on polynomials

1. Suppose $P(X) \in \mathbb{C}[X]$ is such that for every $x \in \mathbb{R}$, $P(x) \in \mathbb{R}$. Show that all the coefficients of $P(X)$ are real numbers.

2. Let $P(X) = X^r + a_1 X^{r-1} + \ldots + a_{r-1}X + a_r$ be a polynomial with complex coefficients such that $P(X)$ divides $X^n - 1$. Show that $|a_i| \leq \binom{r}{i}$.

3. Show that $X^{100} - X - 5$ cannot be written as the product of two polynomials of degree at least one with integer coefficients.

4. Find all complex numbers a, b such that the roots of $x^2 + ax + b$ are $\{a, b\}$.

5. Suppose $P(X)$ is a polynomial such that $P(1) = 0$, $dP/dX(1) = 0$, …, $d^kP/dX^k(1) = 0$. Then show that $(X-1)^k$ divides $P(X)$.

6. Is there a polynomial with integer coefficients which has $\sqrt{2} + \sqrt{3}$ as a root?

7. Find all polynomials $P(X)$ such that $P(P(X)) = X$.

8. Find all polynomials $P(X)$ such that $P(P(P(X))) = X$.

9. Show that for any real number a, b, c,
$$a^2 + b^2 + c^2 \geq ab + bc + ca.$$

10. Show that
$$\sum_{i=0}^{n-1} (i+1)\binom{n}{i+1} 2^i = n3^{n-1}.$$

11. Let $P(X)$ be a quadratic polynomial with real coefficients such that $P(0)$, $P(1)$ and $P(2)$ are all integers. Then show that $P(n)$ is an integer for all integers n.

    Also show that there are quadratic polynomials $P(X)$ such that $P(0)$, $P(1)$ and $P(3)$ are integers, but there are integers n for which $P(n)$ is not an integer.

12. $P(X)$ is a polynomial such that $P(1) = 1$ and $P(-1) = 2$. What is the remainder when you divide $P(X)$ by $X^2 - 1$?

13. Show that the polynomial
$$P(X) = X^n + X^{n-1} + X^{n-2} + a_3 X^{n-3} + a_4 X^{n-4} + \ldots + a_n$$
cannot have all its roots real.