Problems

1. For a square matrix $A$, define $\sin A$ by the power series:
\[
\sin A = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} A^{2n+1}.
\]
Prove or disprove: there exists a $2 \times 2$ matrix $A$ with real entries such that
\[
\sin A = \begin{pmatrix} 1 & 1996 \\ 0 & 1 \end{pmatrix}.
\]

2. Let $G$ be a group with identity $e$ and let $\phi : G \to G$ be a function such that $\phi(g_1)\phi(g_2)\phi(g_3) = \phi(h_1)\phi(h_2)\phi(h_3)$ whenever $g_1g_2g_3 = e = h_1h_2h_3$.

Prove that there exists an element $a \in G$ such that $\psi(x) = a\phi(x)$ is a homomorphism (i.e., for all $x, y$, we have $\psi(xy) = \psi(x)\psi(y)$).

3. If $A$ and $B$ are square matrices of the same size such that $ABAB = 0$, then must it be the case that $BABA = 0$?

4. Let $S$ be a set of real numbers which is closed under multiplication. Let $T$ and $U$ be disjoint subsets of $S$ whose union is $S$. Given that the product of any three elements of $T$ is in $T$, and that the product of any three elements of $U$ is in $U$, show that at least one of the sets $T, U$ is closed under multiplication.

5. Let $G$ be a finite set of real $n \times n$ matrices $M_1, \ldots, M_r$ which form a group under matrix multiplication. Suppose that $\sum_{i=1}^{r} Tr(M_i) = 0$, where $Tr$ denotes the trace (sum of the main diagonal).

Prove that $\sum_{i=1}^{r} M_i$ is the $n \times n$ zero matrix.

6. Let $G$ be a finite group of order $n$ generated by $a$ and $b$. Prove or disprove: there is a sequence $g_1, \ldots, g_{2n}$ such that every element of $G$ occurs exactly twice, and $g_{i+1}$ equals $g_ia$ or $g_ib$ for each $i = 1, \ldots, 2n-1$, and $g_1 = g_{2n}a$ or $g_{2n}b$.

7. Let $G$ be a finite group with identity $e$. If $g$ and $h$ are two elements in $G$ such that $g^3 = e$ and $ghg^{-1} = h^2$, then find the order of $h$.

8. Suppose $x, y, z$ are real numbers with $x + y + z = 4$ and $x^2 + y^2 + z^2 = 6$. Show that each of $x, y, z$ lies in the interval $[2/3, 2]$. Can $x$ attain the extreme values $2/3$ and $2$?