Problems on power series

1. Use the technique of generating functions to solve the recurrence relation $a_0 = 1, a_1 = 0, a_2 = -5$ and for $n > 3$,

   $a_n = 4a_{n-1} - 5a_{n-2} + 2a_{n-3}$.

2. Prove that the value of the nth derivative of $x^3/(x^2 - 1)$ at $x = 0$ is 0 for n even, and $-n!$ for n odd.

3. Let p, q with $p \in (0, 1/2]$ be real numbers satisfying $1/p - 1/q = 1$. Prove that $p + p^2/2 + p^3/3 + \cdots = q - q^2/ + q^3/3 - \cdots$.

4. Show that $\sum_{n=0}^{\infty}(\sin n\theta)/n! = \sin(\sin\theta)e^{\cos\theta}$.

5. Let $T_n = \sum_{i=1}^{n}(-1)^{i+1}/(2i - 1)$, and $T = \lim_{n\longrightarrow\infty} T_n$. Show that:

$$\sum_{n=1}^{\infty} T_n - T = \frac{\pi}{8} - \frac{1}{4}.$$

6. Putnam problem: For $0 < x < 1$, express $\sum_{n=0}^{\infty} \frac{x^{2n}}{1-x^{2n+1}}$ as a rational function of x

7. Putnam problem: Define $S_0 = 1$. For $n \geq 1$, let S_n be the number of $n \times n$ symmetric matrices with nonnegative entries, and all row sums equal to 1. Prove:

   (a) $S_{n+1} = S_n + nS_{n-1}$

   (b) $\sum_{n=0}^{\infty} S_n x^n/n! = e^{x+x^2/2}$.

8. Putnam problem: If n is a positive integer, let $(B(n)$ be the number of ones in the base 2 expression for n. For example, $B(6) = 2$, $B(15) = 4$. Determine whether or not $e^{\sum_{n=1}^{\infty} B(n)/n(n+1)}$ is a rational number.

9. Putnam problem: For each positive integer n, let $f_n(x)$ denote the function:

$$f_n(x) = \frac{\sum_{0\leq k\leq n/2}\binom{n}{2k}x^k}{\sum_{0\leq k\leq n/2}\binom{n}{2k+1}x^k}$$

   Express $f_{n+1}(x)$ rationally in terms of $f_n(x)$ and x. Determine $\lim_{n\longrightarrow\infty} f_n(x)$ for all of the values of x that you can.

10. Putnam problem: Let $f_0(x) = e^x$ and $f_{n+1}(x) = xf_n'(x)$ for $n = 0, 1, 2, \ldots$. Show that

$$\sum_{n=0}^{\infty} \frac{f_n(1)}{n!} = e^e.$$