MATH 642:491–Fall, 2012

Problems on Analysis and Geometry

1. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a continuous function such that for every a, b, if $\frac{f(a)+f(b)}{2} = f\left(\frac{a+b}{2}\right)$. Show that $f(x) = \alpha x + \beta$ for some α, β.

   If we drop the condition that f is continous, does the same conclusion hold?

2. Let $x_0 = 1$. For each $n \geq 1$, let $x_n = (\sqrt{2})^{x_{n-1}}$. Show that $\lim_{n\to\infty} x_n$ exists, and find this limit.

3. Suppose $f : \mathbb{R} \to R$ is a n-times differentiable function, and suppose $a_1 < a_2 < \ldots < a_{n+1}$ are such that $f(a_i) = 0$ for each $i \in \{1, \ldots, n+1\}$. Show that there is some $b \in [a_1, a_{n+1}]$ such that $f^{(n)}(b) = 0$.

4. Show that there is a unique real number c such that for every differentiable function $f : [0, 1] \to \mathbb{R}$ with $f(0) = 0$ and $f(1) = 1$, the equation $f'(x) = cx$ has a solution.

5. Find all continuous functions f such that for every $x > 0$,

$$\int_1^x f(t)dt = \int_x^{x^2} f(t)dt.$$

6. Does there exist a collection $\mathcal{F}$ of uncountably many subsets of $\mathbb{N}$ such that for every $A, B \in \mathcal{F}$, either $A \subset B$ or $B \subset A$?

7. If the angle A of a triangle ABC is doubled, but the lengths of the sides AB and AC are kept the same, the area of the triangle ABC stays the same. Find the angle A.

8. Show that $\sqrt{2} + \sqrt{3}$ is irrational. Show that $\sqrt{2} + \sqrt{3} + \sqrt{5}$ is irrational.

   Bonus (very hard!): Show that $\sqrt{2} + \sqrt{3} + \sqrt{5} + \sqrt{7}$ is irrational.

9. In a triangle ABC with side lengths a, b, c, angle A is twice angle B. Show that $a^2 = b(b + c)$.

10. Prove that if one angle of a triangle is equal to 120 degrees, then the triangle formed by the feet of the angle bisectors is right angled.

11. Show that any rectangle inscribed in an ellipse has its sides parallel to the axes of the ellipse (unless the ellipse is a circle).