Homework 3

Due Date: November 16, 2012.

1. Let \( f(c) \) equal the \( c \)-color Ramsey number \( R(3, 3, \ldots, 3) \) (namely, the smallest integer \( n \) such that any \( c \)-coloring of the edges of \( K_n \) contains a monochromatic triangle).

   Show that \( (f(c_1 + c_2) - 1) \geq (f(c_1) - 1) \cdot (f(c_2) - 1) \). Thus show that \( f(c) \geq 5^\lfloor c/2 \rfloor \).

   What upper bound for \( f(c) \) does the proof of the Ramsey theorem from class give? Today it is unknown what the correct behavior for \( f(c) \) is.

2. Let \( \delta \in (0, 1/2) \) be a constant. We want to find a subset \( C \) of \( \{0, 1\}^n \) of size as large as possible such that no two elements have Hamming distance \( \leq \delta n \). This is a discrete “sphere packing” problem. Express all your answers in terms of the binary entropy function \( H \).

   (a) Pick \( C \) to be a random set of size \( K \). How large can you take \( K \) such that \( C \) has the desired property with probability at least \( 1/2 \)?

   (b) Use the method of alterations to find a larger set \( C \) with the desired property. How large can you make \( C \) this way?

   (c) By a volume packing argument, show that no such \( C \) can have \( |C| \geq 2^{(1-H(\delta/2)+o(1))n} \).

   (d) Now we consider a dual covering problem. We want to find a subset \( C \) of \( \{0, 1\}^n \) of size as small as possible such that every element of \( \{0, 1\}^n \) is within Hamming distance \( \epsilon n \) of some element of \( C \).

   (e) How small a \( C \) can you find with this property?

   (f) Show that no such \( C \) can have \( |C| \leq 2^{(1-H(\epsilon)-o(1))n} \).

3. (Simultaneously big cuts): Let \( G_1, G_2, \ldots, G_k \) be graphs with the same vertex set \( V \), edge sets \( E_1, E_2, \ldots, E_k \subseteq \binom{V}{2} \). Show that for each \( \epsilon > 0 \), there exists a \( t \) depending only on \( \epsilon, k \), such that if \( |E_i| > t \) for all \( i \), then there exists a partition \( V = S \cup T \) such that for each \( i \), at least \( (1/2 - \epsilon) \) fraction of the edges in \( E_i \) go between \( S \) and \( T \).

4. Let \( \alpha, \beta \in (0, 1) \) be constants.

   (a) Show that if \( \beta > \alpha^2 \), then there exists a collection of exponentially many subsets of \( [n] \) with size \( \alpha n \) such that any two subsets of the collection have intersection size at most \( \beta n \).

   (b) Show that if \( \beta < \alpha^2 \), then any such collection has size at most \( n^{O(1)} \).