Last step of proof of Baranyai's Theorem

Each $F_i$ is a partition of $[L]$
consisting of $\frac{n}{k}$ parts (possibly repeated)

We know: $\forall S \subseteq [L]$

\[ \# \{ i \mid s \in F_i \} = \binom{n-L}{k-1|s|} \]

Capacity of $s \to F_i = 1$

Capacity of $F_i \to S = 1$ if $S \subseteq F_i$

Capacity of $S \to t = \binom{n-(k+1)}{k-(|s|+1)}$
Max flow ≤ M.

Consider the following flow:

\[ f(s \rightarrow y_i) = 1 \]

\[ f(y_i \rightarrow s) = \frac{k-|S|}{n-l} \]

\[ f(s \rightarrow t) = \frac{n-(l+1)}{k-(|S|+1)} \]

Why is this a flow?

Constraint at \( y_i \):

\( \text{Inflow} = 1 \)

\( \text{Outflow} = \sum_{S \in G} \frac{k-|S|}{n-l} \)

\[ = \frac{1}{n-l} \cdot \sum_{S \in G} (k-|S|) \]

\[ = \frac{1}{n-l} \left( \sum_{S \in G} k - \sum_{S \in G} |S| \right) \]

\[ = \frac{1}{n-l} \left( \frac{n}{k} \cdot k - l \right) \]

\[ = 1 \]
Constraint at \( |S| \).

Inflow = \[ \frac{k - |S|}{n - l} \cdot \binom{n - l}{k - |S|} \]

Outflow = \[ \binom{n - (k+1)}{k - (|S|+1)} = \frac{k - |S|}{n - l} \binom{n - l}{k - |S|} \]

So, it is a flow.

Value = \( M \).

So there is an integral flow with value = \( M \). (1)

That flow gives a choice each \( S \in S_i \), a set \( S \in S_i \) s.t.

each set is chosen \( \binom{n - (k+1)}{k - (|S|+1)} \) times, as we wanted.