Due Date: February 13, 2012.

Some quick reminders about $O$, $o$, $\Omega$, $\omega$, and $\Theta$:

1. We write $f(n) = O(g(n))$ if there are positive constants $c$ and $n_0$ such that $0 \leq f(n) \leq cg(n)$ for all $n \geq n_0$.

2. We write $f(n) = o(g(n))$ if, for any constant $c$, there is a constant $n_0$ such that $0 \leq f(n) \leq cg(n)$ for all $n \geq n_0$.

3. We write $f(n) = \Omega(g(n))$ if there are positive constants $c$ and $n_0$ such that $0 \leq cg(n) \leq f(n)$ for all $n \geq n_0$.

4. We write $f(n) = \omega(g(n))$ if, for any constant $c$, there is a constant $n_0$ such that $0 \leq cg(n) \leq f(n)$ for all $n \geq n_0$.

5. We write $f(n) = \Theta(g(n))$ if there are positive constants $c_1$, $c_2$, and $n_0$ such that $0 \leq c_1g(n) \leq f(n) \leq c_2g(n)$ for all $n \geq n_0$.

6. For every constants $c, d > 0$,
   $$n^d = o(2^{cn}),$$
   $$n^d = \omega(\log^c n).$$

---

Questions

1. For each of the following pairs of functions, indicate whether the function on the left is $O$, $o$, $\Omega$, $\omega$, or $\Theta$ of the function on the right.

   \[
   \begin{array}{cc}
   n^2 & n^3 \\
   \log(n^2) & \log(n^3) \\
   2n & 3n \\
   2 & 3 \\
   2n + \sin(n) & n + 3 \\
   n^2 & 2\sqrt{n} \\
   n^3 \log_2 n & 3\log_2 n + 5 \\
   \log n & n \log \log n \\
   \log n & \log \log(n^n) \\
   2^{\log n} & (\log n)^{10} \\
   2^{2^{\log n}} & 2^n \\
   \end{array}
   \]
2. Show that $2^n = o(n!)$ and $n! = o(n^n)$.

3. If $f(n) = O(\log n)$, then does it follow that $2^{f(n)} = O(n)$? Why or why not?
   If yes, prove it. If not, what should be the condition on $f(n)$ to get this conclusion?

4. (a) Suppose $T: \mathbb{N} \rightarrow \mathbb{R}$ satisfies

   \[
   T(1) = 10 \\
   T(n) \leq T(\lfloor n/3 \rfloor) + n
   \]

   Prove by induction on $n$ that $T(n) \leq 100n$.

   (b) Suppose $T(n) \leq 3T(\lfloor n/2 \rfloor) + n$. Prove by induction on $n$ that $T(n) \leq O(n \log_3 2)$.

5. Given an $n$ digit integer $N$ as input, how quickly can you decide whether $N$ is a perfect power or not (i.e., do there exist integers $M \geq 1$ and $k \geq 2$ such that $N = M^k$)?

6. You are given a two dimensional $k \times (26^k - 1)$ array, and you are told that it contains every possible string of capital letters of length $k$, except for one of them. It is easy to find the missing one by reading all $\Theta(k \cdot 26^k)$ entries of the array. Show that this task can in fact be done while reading only $\Theta(26^k)$ entries of the array.