2.1.5) \(\sqrt{\text{let } G \text{ be a group and } xyz = 1.}\)

To show \(xyz = 1\), multiply \(xyz = 1\) by \(x^{-1}\) on the left side, to yield \(x^{-1}xyz = x^{-1}\) and \(yz = x^{-1}\). Now, we multiply by \(x\) on the right to yield \(yzx = x^{-1}x = 1\), which was to be shown.

However, it does not follow that \(yxz = 1\). Assume that \(yxz = 1\). Then, \(xyz = yxz\), and \(xy = yx\). Therefore, if the group is not Abelian, the claim does not follow. We provide a counterexample.

Let \(G = \mathbb{R}^{n \times n} \) of invertible \(2 \times 2\) matrices with matrix multiplication. Let \(x = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \), \(y = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \), \(z = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \), \(1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \).

Then, \(xy = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\), and \(yx = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\). \((xy)z = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\)

but \((yx)z = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\), and the claim is disproven.

2.1.4) \(\sqrt{\text{let } S \text{ be a set and let } ab = a, \text{ for any } a,b \in S.}\)

Let \(c,d,e \in S\), and we show \((cd)(e) = (c)(de)\)

\((cd)e = (c)e = c\) using the fact that \(ab = a\) for any \(a,b\)

and \((c)(de) = c(c) = c\), thus \((cd)e = c(de)\).
2.1.10) Let $G$ be a group, $a, b \in G$. Let $ax = b$. We show $x$ is unique in $G$.

Assume that $x$ is not unique. Then $ax = b$ and $ay = b$, $x, y \in G$, $x \neq y$. Then, multiply on the left by $a^{-1}$ in each equation to yield $x = a^{-1}b$, $y = a^{-1}b$, thus $x = y$, a contradiction, so $x$ is unique.

We now show $x \in G$. We have $x = a^{-1}b$. As $a \in G$, $a^{-1} \in G$ and $b \in G$, thus $a^{-1}b \in G$, and $x \in G$.

2.2.1) Let $A = (\begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix})$. We shall calculate $A^k$ until we reach $A^2 = (\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix})$.

$$A^2 = (\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix})(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}) = (\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}).$$

$$A^3 = (\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix})(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}) = (\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}).$$

$$A^4 = (\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix})(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}) = (\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}).$$

$$A^5 = (\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix})(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}) = (\begin{smallmatrix} -i & 0 \\ 0 & i \end{smallmatrix}).$$

$$A^6 = (\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix})(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}) = (\begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix}).$$

Therefore, the cyclic group is $\{ (\begin{smallmatrix} 0 & i \\ i & 0 \end{smallmatrix}), (\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}), (\begin{smallmatrix} i & 0 \\ 0 & i \end{smallmatrix}), (\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}), (\begin{smallmatrix} -i & 0 \\ 0 & -i \end{smallmatrix}) \}$.

2.2.2) Let $a, b \in G$. Let $G$ be a group, $a$ have order 5, and $a^2 b = b a^3$.

We show $ab = ba$. Multiply on both sides of the equation by $a^2$ on the left. Then, $a^2 b a^2 = b a^5 = b$. We substitute this into the original equation, so $a^3 b = a^5 (a^3 b a^2) = a^6 b a^2 = a b a^2 = b a^3$. We now multiply by $a^2$ on the right, and we have $a b a^5 = a b a^5 = b a$. So $a b = b a$. 


2.2.11) Let $G$ be a group, $a, b \in G$. We show $ab$ has the same order as $ba$.

Let $k$ be the smallest integer such that $(ab)^k = e$. Then,

$$(ab)^k = a(ba)(ba)\cdots(ba) = a(ba)^k = a(ba)^{k-1}b.$$ 

Then, $a(ba)^{k-1}b = e$, so $(ba)^{k-1} = a^{-1}b^{-1}$. We show $(a^{-1}b^{-1}) = (ba)^{-1}$,

$$(a^{-1}b^{-1})(ba) = a^{-1}(b^{-1}b)a = a^{-1}a = e.$$ 

$$(ba)(a^{-1}b^{-1}) = b(a^{-1})b^{-1} = b b^{-1} = e,$$ 

so $(a^{-1}b^{-1}) = (ba)^{-1}$.

Thus, $(ba)^{-1} = (ba)^{-1}$. We multiply by $(ba)$ on both sides to yield $(ba)^k = e$. We now show this $k$ is the smallest.

Assume $p < k$, such that $(ba)^p = e$. Then, $(ba)^{p+1} = (ba)^{-1}$, and $a(ba)^{p+1}b = e$ and $(ab)^p = e$, which is a contradiction as $k$ is the smallest for $(ab)$ such that $(ab)^k = e$, so $k$ is the smallest for $(ba)$, too, and $(ba)$ is of order $k$, and the result follows.

\[
\sqrt{\text{2.2.17})} \quad \text{Let } G \text{ be a group. Let any element of } G \text{ have order } 2, \text{ except for } e. \text{ We show } G \text{ is abelian.}\]

Let $a, b \in G$. Consider $ab$. Then,

$$(ab)^2 = b(ab)^2a = b(ba)^2a = ba,$$ 

as $ba \in G$ and thus has order 2, so $ab = ba$. 

In order for a permutation to map a set of 4 elements back to itself after 2 iterations, it must in a sense only "move" two at a time. For example, transposing the first and second twice results in just an identity transposition, implying order 2.

Then, define \( f(a, b) \) as transposing \( a \) with \( b \). Then, \( f^2(a, b) = \text{id}(a, b) \), \( \text{id} \) is identity.

So, \( f(1,2), f(1,3), f(1,4), f(2,3), f(2,4), f(3,4) \) are all order 2.

We may also swap 4 elements pairwise, i.e., swap 1 with 2, and 3 with 4. We do this twice and we achieve an identity transposition. Define \( f(a, b), (c, d) \) as swapping \( a \) with \( b \), \( c \) with \( d \). Then

\[
\begin{align*}
&f(1,2), (3,4), \\
&f(1,3), (2,4), \\
&f(1,4), (2,3)
\end{align*}
\]

are of order 2 as well, yielding 9 elements of order 2.

Let \( \phi : G \to G' \) be an isomorphism, \( G, G' \) be groups, \( x \in G \), \( \phi(x) \in G' \). We show the orders of \( x \) and \( \phi(x) \) are equal.

Let \( x \) be of order \( k \). Then, \( x^k = e_G \). Consider \( \phi(x^k) \). As \( \phi \) is an isomorphism, \( \phi(x^k) = \phi(x)^k \). Also, as \( \phi \) is an isomorphism, \( \phi(e_G) = e_{G'} \), so \( \phi(x)^k = \phi(e_G)^k = e_{G'} = [\phi(x)]^k \).

So, now we show \( k \) is the smallest such integer.

Assume \( pk \) is such that \([\phi(x)]^p = e_{G'} \). Consider \( \phi^{-1} \), which is also an isomorphism.
We have \( \Phi'(\Phi(x)) = \Phi'(\Phi(x) \cdot \Phi(x) \cdot \Phi(x) \cdot \ldots \cdot \Phi(x)) \)
\[ = \Phi'(\Phi(x)) \cdot \Phi'(\Phi(x)) \cdot \ldots \cdot \Phi'(\Phi(x)) = x \cdot x \cdot x \ldots x = x^k. \] Also, \( \Phi'(e_a) = e_a, \) so we have \( x^k = e_a, \) a contradiction as \( p \not\mid k \) and \( k \) is the smallest integer such that \( x^k = e_a, \) so \( \Phi(x) \) must be of order \( k, \) and \( \text{order}(x) = \text{order}(\Phi(x)). \)

2.3.6) Consider \( \mathbb{Z}_3 \) with addition mod 3, \( \mathbb{Z}_3 = \{0, 1, 2\}, \) and the group \( G = \{1, x, x^2\}, \) where \( x^2 = 1 \) and the operation of \( \circ, \) where, if \( a \in G, b \in G, \) then \( a \circ b = x^n \) for some \( n, m \) in \( \{0, 2\} \subseteq \mathbb{Z}, \) then \( a \circ b = x^p \cdot x^q = x^{p+q}. \) Then, \( \Phi_1 : \mathbb{Z}_3 \to G, \)
\[ \Phi_1(n) = x^n \] is an isomorphism, and also \( \Phi_2(n) = x^n \) is an isomorphism, yet they are distinct, as the mapping goes:

\[
\begin{align*}
0 & \leftrightarrow 1 \\
1 & \leftrightarrow x \\
2 & \leftrightarrow x^2
\end{align*}
\]

for \( \Phi_1 \) and

\[
\begin{align*}
0 & \leftrightarrow x \\
1 & \leftrightarrow x^2 \\
2 & \leftrightarrow 1
\end{align*}
\]

for \( \Phi_2 \) \text{ is not an isomorphism. Need to map identity to identity.}

2.4.5) Let \( G \) be an abelian group. Define \( \Phi : G \to G \) as \( \Phi(x) = x^n. \) We show \( \Phi \) is a homomorphism.

Let \( a, b \in G. \) Then, we have \( \Phi(a) \cdot \Phi(b) = a^n \cdot b^n = \underbrace{a \cdot a \cdot \ldots \cdot a}_{n \text{ times}} \cdot \underbrace{b \cdot b \cdot \ldots \cdot b}_{n \text{ times}} = (ab)^n \) by commutativity

\( = (ab)^n = \Phi(ab), \) so \( \Phi(ab) = \Phi(a) \cdot \Phi(b), \) and \( \Phi \) is a homomorphism.
Let \( f : \mathbb{R}^+ \to C^2 \), \( f(x) = e^{ix} \). We prove \( f \) is a homomorphism.

Let \( a, b \in \mathbb{R}^+ \). Then, \( f(a+b) = e^{i(a+b)} = e^{ia}e^{ib} = f(a)f(b) \), so \( f \) is a homomorphism.

Consider \( \ker(f) = \{ a \in \mathbb{R}^+ : e^{ia} = 1 \} \). This is only true when \( a \) has the form of \( 2\pi k \), \( k \in \mathbb{Z} \), as \( e^{0 \cdot 2\pi} = 1 \), \( e^{-2\pi i} = 1 \), \( e^{i \cdot c} \), so \( \ker(f) = \{ 2\pi k : k \in \mathbb{Z} \} \).

The image of \( f \) will be \( \{ e^{ix} : x \in \mathbb{R}^+ \} \). As any complex number has the form \( z = re^{i\theta} \), in this case we will have \( r = 1 \), so the image will be a circle centered at \( 0 \) in the complex plane with radius \( 1 \).

2.4.9a) Let \( \Psi \) and \( \Phi \) be homomorphisms. Consider \( \Phi \circ \Psi \).

Assume \( \Psi : G \to H \) and \( \Phi : H \to J \), \( G, H, J \) groups, so that \( \Phi \circ \Psi : G \to J \). Let \( a \in e \in G \). We wish to show \( \Phi \circ \Psi(a) = [\Phi \circ \Psi(a)] \cdot [\Phi \circ \Psi(b)] \). So, \( \Phi \circ \Psi(ab) = \Phi(\Psi(ab)) = \Phi(\Psi(a) \cdot \Psi(b)) \) as \( \Psi \) is a homomorphism.

\( = \Phi(\Psi(a)) \cdot \Phi(\Psi(b)) \) as \( \Phi \) is a homomorphism.

\( = \Phi \circ \Psi(a) \cdot \Phi \circ \Psi(b) \), proving that \( \Phi \circ \Psi \) is a homomorphism.

b) We proceed by definitions. \( \ker(\Phi \circ \Psi) = \{ a \in G : \Psi(\Phi(a)) = e \} \).

This is equivalent to \( \{ a \in G : \Phi(\Psi(a)) \in \ker(\Psi) \} \). As such, if \( a \in \ker(\Psi) \), then \( a \in \ker(\Phi \circ \Psi) \), as \( \Psi(a) = e \), so that \( \Phi(\Psi(a)) = e \). However, it is not necessarily true that \( \ker(\Phi \circ \Psi) \subseteq \ker(\Psi) \) as \( \Psi(a) \) may not be in \( \ker(\Phi) \). All we know is \( \ker(\Phi) \subseteq \ker(\Phi \circ \Psi) \).
2.6.5) Let $H, K \leq G$ be subgroups, with orders 3 and 5. We show $H \cap K = \{e\}$.

Clearly, let $H, K$ be subgroups, so $\{e\} = H \cap K$.

Now, let $x \in H \cap K$. Assume that $x \neq e$ for contradiction. Then, $x \in H \cap K$ implies $x^3 = 1$. However, as $x \in K$, we have that $x^3 \neq 1$, as $x \neq e$ by assumption, and $x$ must be of order 5, so only $x^5 = 1$, $x \neq 2$, and $3 \neq 5k$, so $x^3 = 1 \neq x^3$, a contradiction, so $x = e$ and thus $x \in \{e\}$, so $\{e\} = H \cap K$.

2.6.7(a) Let $G$ be an abelian group of odd order. Let $\varphi(x) = x^2$, $\varphi: G \rightarrow G$. We show $\varphi$ is an automorphism.

Let $x, y \in G$. Then, $\varphi(xy) = (xy)^2 = (xy)(xy) = (x)(y)(y)$ by commutativity, $(xy)(xy) = x^2 y^2 = \varphi(x) \varphi(y)$, so $\varphi$ is a homomorphism.

We show $\varphi$ is a bijection. Let $\varphi(x) = \varphi(y)$, $x, y \in G$.

Then, $x^2 = y^2$. Assume for contradiction that $x \neq y$.

Then, $x^2 (y^2)^{-1} = (xy)^2 = e$. But $x \neq y$, so $xy^{-1} \neq e$, and so $xy^{-1}$ has order 2, which is a contradiction as $G$ is of odd order, so $x = y$ and $\varphi$ is injective.

Let $x \in G$. We show there is a $y \in G$ such that $\varphi(y) = x$.

Let $G$ be of order $n$. Then $n = 2k + 1$, $k \in \mathbb{Z}$, $k \geq 0$. Thus $x^{2k+1} = x$, so $x = x^{2(1)}$. Let $y = x^{k+1}$. Then, $\varphi(y) = \varphi(x^{k+1}) = (x^{k+1})^2 = x^{2k+2} = x$, so $\varphi$ is surjective. $x = x^{2k+2}$ only holds as valid since $G$ is a group of finite order. Thus, $\varphi$ is a homomorphism and bijection between the same group, so it is an automorphism.
2.6.8) Let $W$ be the additive subgroup of $\mathbb{R}^n$ of solutions to $AX = 0$. We show the set of solutions to $AX = B$ is a coset of $W$.

To do this, we must show that this set is equivalent to the set $c + W = \{c + w : c \in \mathbb{R}^n, w \in W\}$. As addition is abelian in vector spaces, $c + W = W + c$. To avoid technicalities, let $v$ be a solution to $AX = B$, so that $Av = B$. Then the set $Z = \{z \in \mathbb{R}^n : z = v + w\}$.

Let $u \in Z$. Then, $Au = A(u + v) = A(u) + Av = B + 0 = B$. So $u \in Z$. You already have $u - v \in W$ by def, so $u \in Z$.

Thus, $u = v + w$, which implies $u - v = w \in W$, so $u \in Z$. Therefore, $Z = W$, and the claim is proven.

2.6.10(a) Let $H$ be a subgroup of index 2, $H \leq G$, $G$ group. Then the group only has two left cosets, $H$ and $G - H$, as cosets partition the group. We show $gH = Hg$.

Let $a \in G$. If $a \in H$, then $aH$ for some $h \in H$, so $aH = \{ah : h \in H\} = \{h \in H : h \in H\} = H = Ha$, so $aH = Ha$.

If $a \notin H$, we have $aH \neq H$. As cosets partition a group, and $H$ is a coset, then $aH$ must be the other coset, so then $aH \neq H$ implies $ah \in H$. By similar logic, $Hg = gH$, so $Ha = G - H$ and $aH = Ha$, so $H$ is a normal subgroup.
Two points \( g \) and \( h \) and \( h' \) and \( g' \) let \( g \in G \), such that \( g = (x, y) \), and \( h = (x', 0) \), \( x > 0 \).

\[
gh = \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} ax & y \\ 0 & 1 \end{pmatrix}
\]

And \( h'g = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} ax & ay \\ 0 & 1 \end{pmatrix} \)

So, \( gh = \begin{pmatrix} ax & y \\ 0 & 1 \end{pmatrix} \), \( a > 0 \).

\( h'g = \begin{pmatrix} ax & ay \\ 0 & 1 \end{pmatrix} \), \( a > 0 \).

We fix \( x, y \) to be an arbitrary point, \( x, y \in \mathbb{R}^2 \) and let \( a \) vary, \( a > 0 \), and we can fully describe a coset.

Thus, given any point \((x, y)\), the left coset \( gh \) will be

That is, it is a ray in the half-plane. The right cosets are also rays, but they extend from the origin and have slope \( y/x \).

In the case that \( x = 0 \) or \( y = 0 \), the left and right coset will simply be single points lying on the \( y \)-axis.
So, the entirety of the left cosets $gH$ will partition the plane into rays originating from the $y$-axis and extending horizontally. The right cosets will partition the plane into rays originating from the origin, and "revolve" in a sense, almost like a second hand going around a clock.

The book's notation was unclear if $x > 0$ for both subgroups or only $H$. If $G$ requires $x > 0$, then all of the above is valid but restricted to the first and fourth quadrants.

2.7.5) Let $H, N \leq G$, $H, N$ subgroups, $N$ normal, $G$ group. We show $HN = NH$, and $HN$ is a subgroup.

Let $x \in HN$. Then, $HN = \{h \cdot n : h \in H, n \in N\}$, so $x = hn$. We want to show $x$ is of the form $nh$. But, since $h \in H$, we have that $h \in G$. Therefore, for any $g \in G$, we will have $gN = Ng$, so $hN = Nh$. Thus, if $x \in HN$, then $x = hn$, so $x \in hN$, and $x \in HN$, which implies $x \in NH$, so $HN \subseteq NH$. By a symmetric argument, $NH \subseteq HN$, so $HN = NH$.

We show $HN$ is a subgroup of $G$. Let $HN$ as $1 \in H$ and $1 \in N$, as they are subgroups, and $1 \cdot 1 = 1$.

Let $x, y \in HN$. Then, $x = hn$, $h \in H$, $n \in N$, $y = h'\cdot n'$, $h' \in H$, $n' \in N$, so then $xy = hh'(h'\cdot n')$. As $N$ is normal, there is $h \in H$ such that $nh' = h_1n$, so $h(hh')(n'n') = (hh')(nn')$, and $hh' \in H$ and $nn' \in N$ as they are subgroups, so $hh'nn' \in HN$, so $xy \in HN$. 
let \( a \in H \cap N \). Then, \( a = hn, h \in H, n \in N \). As \( H, N \) are subgroups, \( h^{-1} \in H \) and \( n^{-1} \in N \), so \( (hn)^{-1} = n^{-1}h^{-1} \in NH \), but \( NH = H \cap N \), so \( n^{-1}h^{-1} \in H \cap N \), and \( (a)^{-1} = (hn)^{-1} = n^{-1}h^{-1} \in H \cap N \), and thus \( H \cap N \) is a subgroup.

2.8.5) let \( G_1 \) and \( G_2 \) be two infinite cyclic groups. Let \( g_1 \) generate \( G_1 \) and \( g_2 \) generate \( G_2 \), i.e.,

\[ \langle g_1 \rangle = G_1, \quad \langle g_2 \rangle = G_2. \]

Then, the Cartesian product is ordered pairs of the form \( (g_1', g_2') \), \( g_1' \in G_1, g_2' \in G_2 \). In order for any element to be generated in \( G_1 \times G_2 \), we must use the generator \( g_1 \), so \( \langle (g_1', g_2') \rangle \) is the only possible generator of the group.

Then, it must be true that \((e_1, g_2')\) is generated by \((g_1, g_2)\), and \( g_2' \neq e_2 \), where \( e_1, e_2 \) are the identities. So, we must have a \( k \) such that \((g_1, g_2)^k = (e_1, g_2')\). But, based on the definition of the product multiplication of two groups,

\[ (g_1, g_2)^k = (g_1^k, g_2^k) = (e_1, g_2') \]

So, \( g_1^k = e_1 \) implies \( k = 0 \), as \( g_1 \) is infinite order. But, \( g_2^0 = e_2 \neq g_2' \), so the element \((e_1, g_2')\) cannot be generated from \((g_1, g_2)\). We can use a similar argument to show \((g_1', g_2), (g_1, g_2')\) and \((g_1', g_2')\) cannot be used as generators. As these are the only four choices and they fail, there is no generator for \( G_1 \times G_2 \), so it is not cyclic.
2.5. (a) Let $x \in G$ have order $m$, and $y \in G'$ have order $n$. Then, we want $k$, such that $(x,y)^k = (e_1,e_2)_k$, where $k > 0$ is the smallest possible integer. Then, $(x,y)^k = (x^k, y^k) = (e_1, e_2)$ implies $x^k = e_1$ and $y^k = e_2$. So, $x^k = x^{am}$, $y^k = y^{bn}$, $a, b \in Z_4$, $a, b > 0$. So, this amounts to finding the smallest $a$ and $b$ such that $am = bn$, or equivalently, the least common multiple of $m$ and $n$. Thus, the order of $xy$ is the LCM of $m$ and $n$.

(5) (a) For this problem, a permutation will be written as $(a, b, c, \ldots k)$, so $f(1) = a$, $f(2) = b$, \ldots $f(n) = k$. E.g., $(213)$ means $f(1) = 2$, $f(2) = 1$, $f(3) = 3$.

We show $H$ is a subgroup. The identity function $e$ in $H$ as $(123, \ldots n)$ has $f(1) = 1$. If $f, g \in H$, then $f(1) = 1$, $g(1) = 1$, so $(fog)(1) = f(g(1)) = f(1) = 1$. So, $fog$ is in $H$, let $f, g \in H$. So, $f(1) = 1$. Then, $f^{-1}(f(1)) = f^{-1}(1)$, and $f^{-1}(1) = 1$, so $f^{-1} \in H$, and $H$ is a subgroup.

(b) Consider any $g \in S_n$, so that $g = (g_1, g_2, \ldots g_n)$. Then, $g \in H = \{e, g, g^2, \ldots g^{n-1}\}$, where $f(1) = 1$, which means $(g^i) = g_1$, but the other $f(i), i > 1$, may permute as they wish. So, if we have a function $g$, its left coset $gH$ is

$\{eg, eg^2, \ldots g^{n-1}\}$

So, in essence, we are considering all the permutations where $f(i) = k$, $1 \leq k \leq n$. 

Thus, the left cosets will be of the form

\[ gH = \{ \bar{f} \in S_n : f(i) = k, 1 \leq k \leq n, g(0) = k \} \]

If a function \( f \) is a bijection where \( f(1) = g_1 \), and also \( g(1) = g_1 \), then we also have \( fH = gH \). We can construct a permutation \( f \) such that \( gH = \Psi \); such that \( f \) "rearranges" \( g \) for its final \( n-1 \) elements, yet still has \( f(1) = g_1 \), so \( \bar{f} = gH \) for some \( f \), and \( gH = gH \), implying \( \Psi gH = gH \), so \( \Psi H = gH \).

(3) In this case, consider \( h \circ g \), where \( h(0) = 1 \).
As \( g \) is a bijection, we can find \( g^{-1}(1) \). This is not necessarily the first element in the permutation.
We may have \( g^{-1}(1) = 1, g^{-1}(4) = 2, \) etc. Assume \( g^{-1}(i) = k, 1 \leq k \leq n \).
Then, \( g(k) = 1 \) as it is a bijection, so

\[(h \circ g)(k) = h(g(k)) = h(1) = 1\]

So, in the left cosets, we are focused on the first element, in the permutation, i.e., \( f(1) \), and let it vary from 1 to \( n \). In the right cosets, we vary from the first element to the last element, letting \( f(k) = 1 \) in each case. In set notation, if \( g^{-1}(i) = 1 \),

\[Hg = \{ \bar{f} \in S_n : f(1) = 1, f(i) = k, 1 \leq k \leq n, g(k) = 1 \}\]

So, to summarize, the left cosets are permutations where we fix the first value \( f(0) \), and let it vary from 1 to \( n \), and the right cosets, for the value 1, and vary along each \( f(k) \) in the permutation, considering \( f(1) = 1, f(2) = 1, f(3) = 1, \) etc., and consider the set of permutations in each case.
Let $n \geq 3$. Then, $H$ is not a normal subgroup.
To do this, we can provide a counterexample for every $k \leq n$. Let $g = (3 \ 1 \ 2 \ 4 \ 5 \ \ldots \ n)$
Then, $g^{-1} = (2 \ 3 \ 1 \ 4 \ 5 \ \ldots \ n)$

Now, consider the function $f = (1 \ 3 \ 2 \ 4 \ 5 \ \ldots \ n) \in H$.
Then, to be more concise, only consider the first 3 values in each function (as these are the only ones rearranged), and just assume 4, 5, \ldots, $n$ is appended in each case, i.e., $(1 \ 3 \ 2) z = (1 \ 3 \ 2 \ 4 \ 5 \ \ldots \ n)$

So, $g^{-1} f g = (2 \ 3 \ 1)(1 \ 2 \ 3)(3 \ 12)$

$= (2 \ 3 \ 1)(2 \ 1 \ 3)$

$= (3 \ 2 \ 1)$

$= (3 \ 2 \ 1 \ 4 \ 5 \ \ldots \ n)$

But, $(g^{-1} f g)(1) = 3$, so $g^{-1} f g \not\in H$, and so $H$ is not normal.
2.10.5) We show \( \mathbb{R}^* / P \) is isomorphic to \( \mathbb{Z}_2^+ \). Define a homomorphism \( f: \mathbb{R}^* \to \mathbb{Z}_2^+ \) as:

\[
    f(x) = \begin{cases} 
        0, & \text{if } x > 0 \\
        1, & \text{if } x < 0
    \end{cases}
\]

We prove \( f \) is a homomorphism.

Let \( a, b \in \mathbb{R}^* \). There are four cases to consider:

- If \( a > 0, b > 0 \), then \( f(ab) = 0 = 0 + 0 = f(a) + f(b) \).
- If \( a > 0, b < 0 \), then \( f(ab) = 1 = 0 + 1 = f(a) + f(b) \).
- If \( a < 0, b > 0 \), then the previous calculation proves this case.
- If \( a < 0, b < 0 \), then \( f(ab) = 0 = 1 + 1 = f(a) + f(b) \),

so \( f \) is a homomorphism.

Now, the kernel of \( f \) is the set of reals that get mapped to \( 0 \), which is precisely the set \( \{ x \in \mathbb{R} : x > 0 \} \) which is \( P \). Thus, by the first isomorphism theorem,

\[
    \mathbb{R}^* / P \cong \mathbb{Z}_2^+
\]

The only technicality to be addressed is that \( f \) is surjective, which it clearly is, as \( f(1) = 0 \) and \( f(-1) = 1 \), and \( \mathbb{Z}_2 = \{ 0, 1 \} \). So, we are done.
2.10.8) Let $H \subseteq \text{GL}_n(\mathbb{R})$ where $H$ is the set of matrices with positive determinants. We prove $H$ is a normal subgroup.

We first prove $H$ is a subgroup. The identity matrix is in $H$, as $\det(I_n) = 1$. Let $A \in H$. Let $\det(A) = r$, $r > 0$. Then $A^{-1}$ exists and is in $\text{GL}_n(\mathbb{R})$. We have $\det(A^{-1}) = \det(A)^{-1} = \frac{1}{r} > 0$, so $A^{-1} \in H$. Let $A, B \in H$. Then, $\det(AB) \geq 0$ and $\det(B) \geq 0$. So, $AB$ exists and is in $\text{GL}_n(\mathbb{R})$. We have $\det(AB) = \det(A) \det(B) > 0$, so $AB \in H$, and so $H$ is a subgroup.

We show $H$ is normal. Let $A \in H$. We show for any $B \in \text{GL}_n(\mathbb{R})$, $BAB^{-1} \in H$. The proof is immediate as $\det(BAB^{-1}) = \det(B) \det(A) \det(B^{-1}) = \det(A) > 0$, so $H$ is a normal subgroup.

We show $\text{GL}_n(\mathbb{R})/H$ is isomorphic to $\mathbb{Z}_2$. Define $g : \text{GL}_n(\mathbb{R}) \to \mathbb{Z}_2$ as

$$g(A) = \begin{cases} 0, & \text{if } \det(A) > 0 \\ 1, & \text{if } \det(A) < 0 \end{cases}$$

The proof that $g$ is a homomorphism is identical to the proof in 5. $g$ is surjective since $\det(E^{11}) = 0$ and $\det(E^{11}) = 1$. The kernel of $g$ will be any non-matrix whose determinant is positive, which is exactly $H$, so by the first isomorphism theorem

$$\text{GL}_n(\mathbb{R})/H \cong \mathbb{Z}_2$$
We first show that $G \times 1$ is a subgroup of $G \times G'$. $(1, 1) \in G \times 1$, so it contains the identity. Let $(a, 1), (b, 1) \in G \times 1$. Then, $(a, 1) \cdot (b, 1) = (ab, 1)$, and $a, b \in G$, so $(ab, 1) \in G \times 1$. Let $(c, 1) \in G \times 1$. Then, $(c, 1)^{-1} = (c^{-1}, 1) \in G \times 1$, since $c \in G$, and $(c, 1) \cdot (c^{-1}, 1) = (1, 1)$, and $(c^{-1}, 1) \cdot (c, 1) = (1, 1)$.

We show $G \times 1$ is normal. Let $(g, 1) \in G \times 1$, let $(h, h') \in G \times G'$. Then, $(h, h') \cdot (g, 1) \cdot (h, h')^{-1} = (hgh^{-1}, h' \cdot h^{-1}) = (hgh^{-1}, 1)$. Since $G$ is a group, $hgh^{-1} \in G$, so $(hgh^{-1}, 1) \in G \times 1$, so $G \times 1$ is a normal subgroup.

Now that $G \times 1$ is established as a group, we can define the group homomorphism $f : G \times 1 \rightarrow G$ as

$$f((g, 1)) = g$$

It is a homomorphism as $f((a, 1) \cdot (b, 1)) = f((ab, 1)) = ab$ $= f((a, 1)) \cdot f((b, 1))$ for any $(a, 1), (b, 1) \in G \times 1$. $f$ is onto since for any $g \in G$, $f((g, 1)) = g$, and $f$ is injective as $f((x, 1)) = f((y, 1))$ implies $x = y$, so $(x, 1) = (y, 1)$. Thus, $f$ is a bijection and an isomorphism, so $G \times 1$ is a normal subgroup isomorphic to $G$.

Now, we show $(G \times G') / (G \times 1) \cong G'$. Define the function $\Phi : G \times G' \rightarrow G'$ as $\Phi((g, g')) = g' \cdot \Phi((g, g'))$, let $(g, g'), (h, h') \in G \times G'$. Then, $\Phi((g, g') \cdot (h, h')) = \Phi((gh, g'h')) = g' \cdot h' = \Phi((g, g')) \cdot \Phi((h, h'))$, so $\Phi$ is a homomorphism. The kernel of $\Phi$ will be

$$\ker(\Phi) = \{(g, 1) : g \in G'\}$$

But this is precisely $G \times 1$. As $\Phi$ is surjective (since for any $g' \in G'$, $\Phi((1, g')) = g'$), by the first isomorphism theorem, $(G \times G') / (G \times 1) \cong G'$. 
2.10.10) Define the map \( f : \mathbb{C}^* \to \mathbb{P} \) as
\[
f(z) = |z|
\]

\( f \) is a homomorphism as, if \( z, w \in \mathbb{C} \), then
\[
f(z \cdot w) = |zw| = |z| \cdot |w| = f(z) \cdot f(w).
\]
Thus, the kernel of \( f \) will be any complex number \( z \) where \( |z| = 1 \), since 1 is the multiplicative identity in \( \mathbb{P} \). Thus, \( \ker(f) = 0 \). \( f \) is surjective. Let \( x \in \mathbb{P} \). Write \( x = a + bi \). Then,
\[
|z| = \sqrt{a^2 + b^2}, \quad \text{so let} \quad c = \frac{a}{|z|} \quad \text{and} \quad d = \frac{b}{|z|}.
\]
Then,
\[
|z| = \sqrt{c^2 + d^2} = \sqrt{x^2} = x.
\]
So, \( f \) is surjective, and we have that, by the first isomorphism theorem,
\[
\frac{\mathbb{C}^*}{\ker(f)} \cong \mathbb{P}
\]

Likewise, define \( g : \mathbb{C}^* \to \mathbb{U} \) as
\[
g(z) = \frac{z}{|z|}
\]

\( g \) is a homomorphism. Let \( z, w \in \mathbb{C}^* \). Then,
\[
g(z \cdot w) = \frac{zw}{|zw|} = \frac{z}{|z|} \cdot \frac{w}{|w|} = g(z) \cdot g(w).
\]
g is also surjective. Let \( u \in \mathbb{U} \). Write \( u = c + di \). Then, \( \sqrt{c^2 + d^2} = 1 \). Thus, \( \) simply map \( u \) to \( u \), since
\[
g(u) = \frac{u}{|u|} = \frac{c + di}{\sqrt{c^2 + d^2}} = \frac{c + di}{1} = c + di = u.
\]
Thus, \( g \) is a surjective homomorphism. The kernel of \( g \) will be the set of complex numbers that map to \( 1 + 0i \). Since any complex number with a non-zero imaginary component will map to a non-zero imaginary component, the kernel must only contain real numbers. Let \( x \in \mathbb{R} \). Then, if \( x < 0 \), \( g(x) = \frac{x}{|x|} = \frac{x}{-x} = -1 \). If \( x > 0 \), then \( g(x) = \frac{x}{x} = 1 \), so the kernel will be the positive reals or \( \mathbb{P} \). So, by the first isomorphism theorem, \( \mathbb{C}^*/\ker(g) \cong \mathbb{U} \).
As isomorphism of groups is an equivalence relation, it suffices to show $\mathbb{R}^+/2\pi \mathbb{Z}^+ \cong U$ and $\mathbb{R}^+/2\pi i \mathbb{Z}^+ \cong U$, where $U$ is the subgroup of complex numbers with absolute value 1 under multiplication.

Define $f: \mathbb{R}^+ \to U$ as $f(\theta) = e^{2\pi i \theta}$, $\theta \in \mathbb{R}$. Then, $f$ is a homomorphism since, if $\theta, \phi \in \mathbb{R}$, then

$$f(\theta + \phi) = e^{2\pi i (\theta + \phi)} = e^{2\pi i \theta} e^{2\pi i \phi} = f(\theta) \cdot f(\phi),$$

so $f$ is also surjective, since if $z \in U$, then $z = 1 \cdot e^{2\pi i \theta}$ for some $\theta \in \mathbb{R}$, so $f(\theta) = z$. Thus, the function $f$ defines a surjective homomorphism. Then, the kernel of $f$ will be all $\theta \in \mathbb{R}$ such that $e^{2\pi i \theta} = 1$. But, $e^{2\pi i \theta} = 1$ exactly when $\theta$ is an integer, since $e^{2\pi ik} = 1$ for all $k \in \mathbb{Z}$. Thus, $\ker(f) = \mathbb{2Z}^+$, and by the first isomorphism theorem,

$$\mathbb{R}^+/\mathbb{2Z}^+ \cong U$$

We define similarly $g: \mathbb{R}^+ \to U$ as $g(\theta) = e^{i\theta}$. $g$ is a homomorphism, which is identical to the above calculation, and $g$ is surjective. Let $z \in U$. Then, $z = e^{2\pi i \theta}$, $\theta \in \mathbb{R}$. So, $g(2\pi \theta) = e^{2\pi i \theta} = z$. Now, the kernel of $g$ will be the real numbers $\theta$ such that $e^{i\theta} = 1$. This occurs exactly when $\theta$ is an integer multiple of $2\pi$, as $e^{2\pi ik} = 1$, $k \in \mathbb{Z}$. Therefore, $\ker(g) = 2\pi \mathbb{Z}^+$. So, we have

$$\mathbb{R}^+/2\pi \mathbb{Z}^+ \cong U$$

As isomorphism is transitive, we have $\mathbb{R}^+/2\pi \cong \mathbb{R}^+/2\pi i \mathbb{Z}^+$.
5.3.1) \[ x^2y \cdot xy^{-1} \cdot x^3y^3 \]

We have \( yx = xy^{-1} \), and also \( xy = yx^{-1} \)

\[ x^7y \cdot xy^{-1} \cdot x^3y^3 = x^2xyy^{-1} \cdot x^2y^3 = x^7y^3 = x^6y^3 \]

\[ x^6y^3 \]

5.3.2) Since \( D_4 \) has order 8, every subgroup must be of order 1, 2, 4, or 8. Denote \( \beta \) as a reflection, \( \alpha \) as a rotation.

There are 10 subgroups we list.

Order 1 Subgroups
\[ \{e\} \] and it is trivially normal.

Order 4 Subgroups
Since any subgroup here will have index 2, they will be normal.

\[ \{ e, \alpha, \alpha^2, \alpha^3 \beta \} = \langle \alpha \rangle \]
\[ \{ e, \beta, \beta \alpha, \beta \alpha^2 \} = \langle \beta, \alpha \rangle \]
\[ \{ e, \beta \alpha, \beta \alpha^2, \beta \alpha^3 \} = \langle \beta, \alpha^2 \rangle \]

Order 2 Subgroups
\( D_4 \), which is a normal subgroup of \( D_4 \).

Order 8 Subgroups
Since \( \beta \alpha = (\beta \alpha)^{-1} \), it is easy to generate the subgroups.

\[ \{ e, \alpha^2 \beta \} = \langle \alpha^2 \rangle \]
\[ \{ e, \beta \alpha^2 \} = \langle \beta \alpha^2 \rangle \]
\[ \{ e, \beta \alpha \} = \langle \beta \alpha \rangle \]
\[ \{ e, \beta \alpha^2 \} = \langle \beta \alpha^2 \rangle \]
and finally \( \{ e, \beta \} = \langle \beta \rangle \).
Of these five subgroups, only $\{e, a^2\}$ is normal. Note that left multiplication by $a$ will create different cosets than right multiplication, as shown below.

$$a \cdot \{e, a^2\} = \{e, a^2\}, \quad a \cdot \{a, a^3\} = \{a, a^3\}$$

$$e \cdot \{e, a^2\} = \{e, a^2\}, \quad e \cdot \{a, a^3\} = \{a, a^3\}$$

$$e \cdot \{e, a^2\} = \{e, a^3\}, \quad e \cdot \{a, a^3\} = \{e, a^2\}$$

$$a \cdot \{e, a^2\} = \{e, a^3\}, \quad a \cdot \{a, a^3\} = \{e, a^2\}$$

$$e \cdot \{e, a^3\} = \{e, a^2\}, \quad e \cdot \{a, a^2\} = \{a, a^3\}$$

$$e \cdot \{e, a^3\} = \{e, a^3\}, \quad e \cdot \{a, a^2\} = \{e, a^2\}$$

$$a \cdot \{e, a^3\} = \{e, a^2\}, \quad a \cdot \{a, a^2\} = \{e, a^3\}$$

$$e \cdot \{e, a^2\} = \{e, a^2\}, \quad e \cdot \{a, a^3\} = \{e, a^3\}$$

But, $\{e, a^2\}$ is normal due to the fact that $a^2 = (a^{-1})^{-1}$, so it commutes with any element in $D_4$ since $a^2 = a^2 b$ and $a^n a^2 = a a^2$, and so it forms a normal subgroup.

(5.3.4a) Let $g \in D_4$. Then, $g = a^n$ or $g = b a^n$, $1 \leq n \leq 9$. If $g = e$, then $g \cdot H = H$.

If $g = a^n$, then $a^n \cdot H = \{a^n, a^n a^2, a^n a^3, a^n a^4, a^{n+5} a^2, a^{n+5} a^3, a^{n+5} a^4\}$. If $n \geq 5$, then $a^n H = \{a^n, a^{n-5} a^2, a^{n-5} a^3, a^{n-5} a^4, a^{n-5} a^5 a^2, a^{n-5} a^5 a^3, a^{n-5} a^5 a^4\}$. Thus, the left cosets are

$$\{e, a^2, a^3, a^4, a^5, a^6, a^7, a^8\}$$

$$\{e, b a^2, b a^3, b a^4, b a^5 a^2, b a^5 a^3, b a^5 a^4, b a^5 a^5 a^2\}$$
b) Define \( f: D_6 \to D_5 \) as
\[
f(\rho_i^0, \alpha_j^0) = \rho_i^0 \alpha_j^0, \quad 0 \leq i < 2, \quad 0 \leq j < 9
\]

Where \( \rho_i, \alpha_j \) are the rotation and reflection actions in \( D_6 \), and likewise for \( \rho_i \alpha_j \) in \( D_5 \). Then, \( f \) is a homomorphism of groups. Let \( \rho_i^0, \alpha_j^0 \in D_6 \). Then
\[
f(\rho_i^0, \alpha_j^0) = f(\rho_i^0, \alpha_j^0) = \left\{
\begin{array}{ll}
\rho_i^0 \alpha_j^0 & \text{if } j = 0 \\
\rho_i^0 \alpha_j^0 & \text{if } j = 1
\end{array}
\right.
\]

But,
\[
\rho_i^0 \alpha_j^0 = \rho_i^0 \rho_j^0 \alpha_j^0 = \rho_i^0 \rho_j^0 \alpha_j^0 = f(\rho_i^0 \rho_j^0) = f(\rho_i^0) f(\rho_j^0) \quad \text{if } j = 0
\]
\[
\rho_i^0 \alpha_j^0 = \rho_i^0 \rho_j^0 \alpha_j^0 = \rho_i^0 \rho_j^0 \alpha_j^0 = f(\rho_i^0) f(\rho_j^0) \quad \text{if } j = 1
\]

So, \( f \) is a homomorphism of groups, \( f \) is also surjective, as if \( \alpha_5 \in D_5 \), the \( f(\alpha_5 \rho_0^0) = \alpha_5 \rho_0^0 \). So, \( f \) is a surjective homomorphism. For the kernel of \( f \), we require that \( i = 0 \). For \( j \), however, since \( e = \alpha_5 \rho_0^0 = \alpha_5 \rho_3^0 \) in \( D_5 \), we have that if \( j = 5k, \ k \in \mathbb{Z} \), then \( \alpha_j = e \) in \( D_5 \). Since the only \( \alpha_j \in D_6 \) where \( 5j \) are \( \alpha_0 = e \) and \( \alpha_5 \), we have that the kernel of \( f \) is
\[
\ker(f) = \{ e \}, \ e^5 \rho_3 = e
\]

So, by the first isomorphism theorem, \( D_6 / \ker(f) \cong D_5 \).
(c) We invoke the theorem that if $H, K$ are normal, $HK = \mathbb{E}G$, and $HK = G$, then $G \subseteq HK$, if $H, K$ are subgroups of $G$.

Use $H$ defined previously as $H = \mathbb{E}1, \alpha^3$. As $H$ is the kernel of a homomorphism, it is normal. Let $K$ be the group: $D_5$. As $|\alpha^3| = 2$, $K \cap D_5 = \mathbb{E}1$. We must show $D_5$ is a subgroup of $D_10$.

Consider the subgroup of $D_10$

$$\{e, \alpha^2, \alpha^4, \alpha^6, \alpha^8, \beta, \beta \alpha^2, \beta \alpha^4, \beta \alpha^6, \beta \alpha^8\} = J$$

Then, there is the isomorphism $\psi : D_5 \to J$ defined as

$$\psi(\alpha^i \beta^j) = \beta^j \alpha^{2i}$$

The fact that $J$ is also isomorphic to $D_5$ is readily seen by dividing the exponent of any $\alpha$ by 2. Thus, $D_5 \cong J$, $J$ has index 2 so it is normal, and $D_5$ is a normal subgroup of $D_10$.

Now, we show $J \cdot H = D_{10}$. If we multiply $J$ by $\alpha^5$, we generate

$$\alpha^5 J = \{e, \alpha^5, \alpha^7, \alpha^9, \alpha^2, \alpha^6, \beta \alpha^5, \beta \alpha^7, \beta \alpha^9, \beta \alpha^2, \beta \alpha^6\}$$

So, $1 J \alpha^5 J = H \cdot J = \{\beta \alpha^j : 0 \leq i < 2, 0 \leq j < 9\} = D_{10}$

So, by the above theorem, $H \cdot J \cong D_{10}$, and since $D_5 \cong J$, $H \cdot D_5 \cong D_{10}$. 

5.5.3) Let $S$ be a set and $G$ a group that acts on $S$. We show $\sim_S$ (if $s' = g s$ for $g \in G$) is an equivalence relation.

Let $s \in S$. Then, $s \sim_S s$ since $e \cdot s = s$ for any $s \in S$, so $\sim$ is reflexive.

Let $s, s', s'' \in S$ where $s \sim_S s'$. Then, $s' = g s$ for some $g \in G$. Then, we must have $g^{-1} (g s) = g^{-1} (s') = (g^{-1} g) s = s = g^{-1} s''$. As $g^{-1} g \in G$, we have $s \sim_S s''$ so $\sim_S$ is symmetric.

Let $s, t, u \in S$. Let $s \sim t$, $t \sim u$. Then, $t = g s$ and $u = h t = h (g s) = (h g) s$, and $h g \in G$ since $G$ is closed, so as $u = (h g) s$, we have that $s \sim u$, so $\sim$ is transitive and thus $\sim$ is an equivalence relation.

5.5.8a) Let $G = GL_n (\mathbb{R})$, $S = \mathbb{R}^n$.

Let $s \in S = \mathbb{R}^n$, so $s$ is a vector in $\mathbb{R}^n$. Then, $\text{orbit} (s) = \{ g s : g \in \text{GL}_n (\mathbb{R}) \}$. We rewrite this as $\text{orbit} (s) = \{ Ax : A \text{ is an } n \times n \text{ matrix}, s \in \mathbb{R}^n \}$. Then, if $s \neq \mathbf{0}$, we have that $\text{orbit} (s) = \mathbb{R}^n - \{ \mathbf{0} \}$, i.e. $\mathbb{R}^n$ without the zero vector. This is because, given any vector $b \in \mathbb{R}^n$, we can always find an invertible matrix $A$ such that $A x = b$. 


To prove this, we show there is an invertible linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^n$ that sends $x \to b$, $x, b \neq \mathbf{0}$.

Extend $x, b$ to a basis for $\mathbb{R}^n$, so that

$$\beta = \{x_1, x_2, x_3, \ldots, x_n\}$$
$$\delta = \{b, u_2, u_3, \ldots, u_n\}$$

Are both bases for $\mathbb{R}^n$. Define $T : \mathbb{R}^n - \mathbb{R}^n$ as

$$T(x) = b, \quad T(x_i) = u_2, \ldots, T(x_n) = u_n.$$

Then, as $T$ maps from a basis in $\mathbb{R}^n$ to another basis in $\mathbb{R}^n$, this implies $T$ is surjective, so it is injective by the dimension theorem, and so $T$ is a bijection and thus invertible. Thus, $T$ also maps $x$ to $b$, and we have found a suitable transformation. Let $A$ be the matrix representative of $T$, so we have that there is an invertible matrix so that $Ax = b$.

However, we have assumed $x \neq \mathbf{0}$ and $b \neq \mathbf{0}$ (otherwise we could not create a basis). Therefore, $D \in \text{image}(T)$ if we stipulate that $x \neq \mathbf{0}$. Thus, if $s \in \mathbb{R}^n$, we have that if $s \neq \mathbf{0}$

$$\text{orbit}(s) = \{Ax : A \in \text{Gl}(\mathbb{R}), x \in \mathbb{R}^n \neq \mathbf{0}, \quad b \in \mathbb{R}^n : b \neq \mathbf{0}\}$$
$$= \mathbb{R}^n - \mathbf{0}$$

If $s = \mathbf{0}$, then $As = \mathbf{0}$ for any $A$, thus in this case $\text{orbit}(s) = \mathbf{0}$, so $\text{Gl}(\mathbb{R})$ can be decomposed into the two orbits $\mathbb{R}^n - \mathbf{0}$ and $\mathbf{0}$. 
b) Write $e_1$ explicitly in vector form as:

$$e_1 = (1, 0, 0, \ldots, 0)$$

Then, the stabilizer of $e_1$ will be $A \in \text{GL}_n(\mathbb{R})$ such that $A \cdot e_1 = e_1$. Consider the $k^{th}$ row of $A$, where $2 \leq k \leq n$. We must have that

$$A_{k1} \cdot 1 + A_{k2} \cdot 0 + A_{k3} \cdot 0 + \ldots + A_{kn} \cdot 0 = 0$$

so that the $k^{th}$ entry of $A \cdot e_1 = 0$. This implies that $A_{k1} + A_{k2} + \ldots + A_{kn} = 0$, so that $A_{k1} = 0$ for $2 \leq k \leq n$. Now, for the first row, we must have that

$$A_{11} \cdot 1 + A_{12} \cdot 0 + \ldots + A_{1n} \cdot 0 = 1$$

which implies $A_{11} = 1$. These are the only constraints we have on the matrix entries, as any other entries in the matrix not in the first column will be multiplied by 0 and not affect the resulting vector. Thus, the stabilizer of $e_1$ is all matrices $A$ whose first column is $e_1$, or

$$\text{stab}(e_1) = \left\{ A \in \text{GL}_n(\mathbb{R}) : A_{11} = 1, A_{k1} = 0, 2 \leq k \leq n \right\}$$

5.5. 9(a) I guess this question means decompose into orbits? So, let $s \in \mathbb{C}^{2 \times 2}$ and consider

$$\text{orbit}(s) = \left\{ A \cdot s : A \in \text{GL}(\mathbb{C}) \right\}$$

If $s$ is the $0$ matrix, $\text{orbit}(s)$ is the zero matrix in $\mathbb{C}^{2 \times 2}$ trivially.
Let $S$ not be the zero matrix. Let $S$ be an invertible matrix. Thus, $S \in \text{GL}_n(\mathbb{C})$. We show $\text{orbit}(S) = \text{GL}_n(\mathbb{C})$.

Let $S \in \text{GL}_n(\mathbb{C})$ and consider

$$\text{orbit}(S) = \{ A_S : A_S \in \text{GL}_n(\mathbb{C}) \}$$

We show that for any $B \in \text{GL}_n(\mathbb{C})$ there is an $A$ such that $A_S = B$. This is easily shown, as we can compute $A = B S^{-1}$. As $\text{GL}_n(\mathbb{C})$ forms a group, we have that $B S^{-1} \in \text{GL}_n(\mathbb{C})$ so for any $B \in \text{GL}_n(\mathbb{C})$, we can always find a matrix $A$ such that $A_S = B$, thus

$$\text{orbit}(S) = \{ A_S : A_S \in \text{GL}_n(\mathbb{C}) \} = \{ B \in \text{GL}_n(\mathbb{C}) \} = \text{GL}_n(\mathbb{C})$$

if $S$ is an invertible matrix.

Finally, let $S$ be a non-zero, non-invertible matrix. So, consider

$$\text{orbit}(S) = \{ A_S : A_S \in \text{GL}_n(\mathbb{C}) \}$$

Then, let $A_1, A_2, \ldots, A_n$ be elementary matrices that put $S$ into reduced echelon form. As $A_1, \ldots, A_n \in \text{GL}_n(\mathbb{C})$, then $A_1 A_2 \ldots A_n \in \text{GL}_n(\mathbb{C})$ so $A_1 A_2 \ldots A_n S \in \text{orbit}(S)$. Then, $S' = A_1 A_2 \ldots A_n S$ has the form

$$S' = \begin{pmatrix} 1 & a \\ 0 & 0 \end{pmatrix}, \quad a \in \mathbb{C}$$

It is impossible that if $(1, 0) \in \text{orbit}(S)$ then $(b, 0) \in \text{orbit}(S)$, $a \neq b$, as no row operation will turn $(1, 0)$ into $(b, 0)$ and any invertible matrix is the product of elementary matrices, so no invertible matrix can multiply $(1, 0)$ to yield $(b, 0)$. 
Thus, the set of $\mathbb{C}^{2 \times 2}$ can be decomposed into infinitely many orbits, namely
\[ \mathbb{C}^3, \text{GL}_n(\mathbb{C}), \text{and } M \text{ Missed row earlier} \]
where $M$ is matrices row equivalent to $(a \ 0), \ a \in \mathbb{C}$.

5.5.11a) Let $s = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \in \text{GL}_n(\mathbb{R})$. We determine the stabilizer of $s$, $\text{stab}(s) = \{ M \in \text{GL}_2(\mathbb{R}) : MS^{-1} = s \}$. This implies that $MS = sM$, so we write $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}
\]

This implies $b = 2b$, so $b = 0$, and $c = 2c$, so $c = 0$. Thus, it is any matrix of the form $\begin{pmatrix} a & 0 \\ 0 & 2 \end{pmatrix}$, for $a \in \mathbb{R}$. Thus, we have that the stabilizer of $s$ is the set of diagonal matrices in $\text{GL}_n(\mathbb{R})$ of dimension $2$, which makes sense since if $AB$ are diagonal matrices, then $A$ and $B$ commute, so $MSM^{-1} = MM^{-1}s = s$. Thus, $\text{stab}(s) = \{ M \in \text{GL}_2(\mathbb{R}) : M \text{ is diagonal} \}$.

We determine $\text{orbit}(s)$. As $s = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$, we can immediately determine its eigenvalues as 1 and 2. As $\text{orbit}(s) = \{ BSB^{-1} : B \in \text{GL}_2(\mathbb{R}) \}$, we can characterize this set as matrices similar to $s$, thus having the same characteristic polynomial. Therefore,
\[ \text{orbit}(s) = \{ M \in \text{GL}_2(\mathbb{R}) : \text{eigenvalues of } M \text{ are } \lambda = 1, 2 \} \]
5.6.1) Let \( \alpha \) be a coset. We determine its stabilizer. Then, \( \text{stab}(\alpha H) = \{ g \in G : \ g(\alpha H) = (g\alpha)H = \alpha H \} \). Then, \( g\alpha H = \alpha H \) means that \( g \cdot g^{-1} \cdot \alpha \cdot H = H \). This implies that \( g \cdot g^{-1} \cdot \alpha \cdot H \) is an element of \( \alpha H \). Otherwise, if \( g \cdot g^{-1} \cdot \alpha \cdot H \neq \alpha \cdot H \), then the coset \( g \cdot g^{-1} \cdot \alpha \cdot H \neq \alpha \cdot H \). Thus, as \( g \cdot g^{-1} \cdot \alpha \cdot H \) is in \( \alpha \cdot H \), we have \( g \cdot g^{-1} \cdot \alpha \cdot H = \alpha \cdot H \). So, the stabilizer of \( \alpha H \) is

\[
\text{stab}(\alpha H) = \{ g \in \alpha H : g^{-1} \cdot \alpha \cdot H = \alpha \cdot H \}
\]

This can be seen since \( g(\alpha H) = \alpha H \) if and only if \( g \cdot g^{-1} \cdot \alpha \cdot H = \alpha \cdot H \). 

5.6.2) Let \( G \) be a group, \( H \) a cyclic subgroup generated by \( x \in G \). Let \( x \) fix every coset of \( H \) when multiplied. We show \( H \) is normal. Thus, for any \( g \in G \), \( x^g \in \langle x \rangle \), we have that \( x^g \cdot H = x \cdot x \cdot \ldots \cdot x \cdot g \cdot H = x \cdot g \cdot H = g \cdot H \). Thus, \( g^{-1} \cdot x^g \cdot H = H \), so for any \( g \in G \), we have that \( g^{-1} \cdot x^g \cdot e \cdot H = H \), but \( x^g \) describes every member of \( H \), which implies that \( H \) is a normal subgroup.

5.7.1) For a cube, consider one face. Then, the stabilizer of that one face will keep that face in the same position, so we can rotate it clockwise once, twice, or three times, or none at all, so the stabilizer of a face has four rotations. The order of that face will be how many different faces of the cube that face can reach by rotation, which is all 6, so \( 6 \cdot 4 = 24 \) rotational symmetries.

For the tetrahedron, consider one triangular face. The rotations that fix this face, i.e., the stabilizer, will rotate only the vertices of that face, giving 3 rotations. The orbit of that face, i.e., how many other faces it can
Switch with, is all four faces of the tetrachoron, so the orbit has four rotations, so one tetrachoron has $3 \cdot 4 = 12$ symmetries of rotation.

5.8.2) Let $G$ act on $S$ and $H = \{ g \in G : g s = s \text{ for all } s \in S \}$.

We show $H$ is a normal subgroup.

$H$ is a subgroup. $e \in H$, since $e \cdot s = s$ for all $s$.

Let $a, b \in H$. Then $b \cdot (a \cdot s) = a \cdot (b \cdot s) = a \cdot s = s$, so $a b \in H$.

Let $g \in H$. Then, $g s = s$ implies $s = g^{-1} s$ by left multiplication of $g^{-1}$, so $g^{-1} \in H$, so $H$ is a subgroup.

We show $H$ is normal. Let $g \in G$, $h \in H$. We show $g h g^{-1} \in H$.

Then, $(g h g^{-1}) (s) = g (h \cdot g^{-1} s) = g \cdot g^{-1} s$ as $g \cdot s \in S$, and as $h \in H$, we have $h \cdot s = s$ for all $s$.

Thus $h \cdot g^{-1} s = g^{-1} s$, so $g (h \cdot g^{-1} s) = g (g^{-1} s) = (g \cdot g^{-1}) s = e \cdot s = s$, so $g h^{} g^{-1} \in H$, and so $H$ is normal.
EC: Let $\text{GL}_n(\mathbb{R})$ be the general linear group, and let $\text{SL}_n(\mathbb{R})$ be the special linear group. Then, we have the homomorphism

$$f: \text{GL}_n(\mathbb{R}) \to \mathbb{R}^*$$

where $f(A) = \det(A)$, $A \in \text{GL}_n(\mathbb{R})$, $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$.

Then, the kernel of $f$ will be $\text{SL}_n(\mathbb{R})$, as $\det(M) = 1$ if $M \in \text{SL}_n(\mathbb{R})$. By the First Isomorphism Theorem, then, we have

$$\text{GL}_n(\mathbb{R}) / \text{SL}_n(\mathbb{R}) \cong \mathbb{R}^*$$

The structure of $\frac{\text{GL}_n(\mathbb{R})}{\text{SL}_n(\mathbb{R})}$ is all the cosets of $\text{SL}_n(\mathbb{R})$, which will be of the form

$$A \cdot \text{SL}_n(\mathbb{R}) = \{M \in \text{GL}_n(\mathbb{R}) : \det(M) = \det(A)\}$$

That is, it partitions $\text{GL}_n(\mathbb{R})$ by the value of its determinant. Thus, the union operation of the cosets, will generate another coset, defined as the following

$$(A \cdot \text{SL}_n(\mathbb{R})) \cdot (B \cdot \text{SL}_n(\mathbb{R})) = \{M \in \text{GL}_n(\mathbb{R}) : \det(M) = \det(AB) = \det(A)\det(B)\}$$

And from this set definition, we can see the isomorphic nature it has with $\mathbb{R}^*$, as multiplying two cosets together is the same as the product of the determinants, or the product of two real numbers.