Numerical Analysis I
Math 373

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About the course

MATH 373 NUMERICAL ANALYSIS I
SUMMER 2018
640:373, Index: 00279, Section H1

Instructor: Surya Teja Gavva
Email: suryateja@math.rutgers.edu
Lectures: MTWH 10:10 am - 12:10 pm, SEC 211, Busch Campus
Course Site: Sakai (Please check for updates regularly)
Office Hours: Tuesday, Thursday 12:15-2:00 in my office Hill 608

About the course

Prerequisites: CALC4 and familiarity with a computer language (Not essential). Basic knowldedge of linear algebra

Textbook: Numerical Analysis, Burden & Faires,
Cengage 978-1305253667 10th Edition, 2015
I will post other important resources on Sakai.

Programming

Programming experience is desirable. There will be computational assignments to implement the numerical algorithms. Could be done using a calculator. But it is much easier and better to learn to write programs. I will post relevant resources for coding in Matlab. You can program in the language of your choice.

Assignments and Exams

The breakdown of grades for the course is as follows:

Quizzes	10%
Homework	25%
Midterm	25%
Final	40%

Quizzes

- We will have short quizzes almost every Tuesday and Thursday.
- The quizzes will mostly be comprised of one calculation problem and one proof.
- There will be 10 quizzes.
- Your lowest 2 quiz grades will be dropped.
- There will be no make-up quizzes.

Homework:

- There will be homework due every Monday and Thursday
- Will be posted on Sakai in the resources section.
- Homework will be due in two parts.
- Part 1 will consist of computational problems and will be graded by a grader.
- Part 2 will consist of more proof based problems and will be graded by me.
- They must be **turned in separately** since they will go to two different people.
- The lowest homework grade for Part 1 and the 2 lowest homework grades for Part 2 will be dropped.
- Late homework will not be accepted; if you know you cannot make class you may email me your homework. The assignment must be in my email mailbox by 10:15 am the day it is due.

Exams

Midterm: There will be one midterm on Wednesday, 25th July in class.

Final: The final (comprehensive) will be on the last day of classes: Wednesday, August 15. It will start at the beginning of class (10:10 am) but it will last THREE HOURS ending at 1:10 pm. The room is TBA.

Make-ups will only be given in genuine, unavoidable and clearly documented instances.

This means there must be a doctor's note, notice of court appearance, etc indicating that you were unable to attend the exam. You should contact me as soon as you know you will not be able to attend the exam.

Topics

- Root Finding
- Polynomial Interpolation
- Numerical Integration
- Numerical Differentiation
- Initial-Value Problems for Ordinary Differential Equations

What is the course about?

Numerical analysis is about numerical algorithms. We want to compute "good" approximations to various quantities of interest. Exact values are often hard to compute because of the complexity of the systems. Exact computations need infinite precision, memory and time resources on a computer. So we settle for the best approximations that we can get with limited resources.
So, in this course we study various numerical methods used in different areas of science.
How to implement these methods? How good are these methods? (Stability, accuracy, rate of convergence etc)

Applications

Numerical computations are the only way to go when the equations involved are so complex to solve analytically. Hence different numerical methods are applied to study many systems in science. The choice of numerical methods to apply depends on the specific problem and studying these methods will help us choose a "good" method that works for the problem at hand.

Applications: Physics:

Model and simulate a physical system. We need to do numerical computations to match the data with experimental observations.
Planetary motion:

$$\frac{d^2\vec{r}}{dt^2} = -\frac{GM}{r^2}$$

Pendulum:

$$\frac{d^2\theta}{dt^2} + \frac{g}{\ell}\sin\theta = 0$$

Heat Flow:

$$\frac{\partial u}{\partial t} - \alpha\nabla^2 u = 0$$

Quantum mechanics/Schrodinger Equation:

$$\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) + V(x)\psi(x) = E\psi(x)$$

Applications: Physics

Maxwell's equations:

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$$

$$\nabla \times \mathbf{B} - \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} = \mu_0 \mathbf{J}$$

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$$

$$\nabla \cdot \mathbf{B} = 0$$

Applications

Biology: Study the genetic evolution(mutations etc) , enzyme production/kinetics, diffusion of virus epidemics etc
Finance: Model and predict the highly fluctuating market trends. Optimally pricing the options
Climate: Predict wind direction, humidity, temperature, pressure etc at a place in future times from the current data
Space missions: Predict the trajectory of the spacecrafts
Many many more– numerical computations are everywhere!

Preliminaries

Please review the basic notions from Calculus

Limits: What is a limit? Continuity? Intermediate Value Property

Differentiation: What is a derivative? Rolle's theorem, Mean value theorem, Taylor approximation

Integrals: Riemann integrals, antiderivatives, Fundamental theorem of calculus

Ordinary Differential Equations:

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  $|\sin x - p(x)| \le 0.00001$ for $x \in [0, \pi]$
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  Calculate $\int_0^1 \sqrt{\sin x} dx$ to 10 decimal values.

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- Numerical Integration
  Calculate $\int_0^1 \sqrt{\sin x} dx$ to 10 decimal values.
- Numerical Differentiation
  How do we calculate the derivatives?
- Initial-Value Problems for Ordinary Differential Equations
  How can we solve all the differential equations we saw before?

Today

We study root finding algorithms.
Problem: Find numerical approximation to roots of the equation $f(x) = 0$
We will discuss iterative algorithms.
Questions:
When do the iterates converge to the root?
Local convergence–If we start with an initial guess *sufficiently(?)* close to the roots, they converge to the roots.
How fast do the iterate converge?
At what rate $|x - x_n|$ goes to zero with $n \to \infty$

Bisection method

Bisection method

Idea: If a, b are such that $f(a)$ and $f(b)$ have opposite sign, there is a zero of f in the interval (a, b)

Iterate this by bisecting the interval–find the subinterval which has the root (how?)

Continue.

Bisection method

Set $a_0 = a$ and $b_0 = b$. For $n = 0, 1, \cdots$,
Set $x_n = \frac{a_n + b_n}{2}$.
If $f(a_n)f(x_n) < 0$, set $a_{n+1} = a_n, b_{n+1} = x_n$.
If $f(x_n)f(b_n) < 0$, set $a_{n+1} = x_n, b_{n+1} = bn$.
A root always lies in $[a_n, b_n]$ and the length of the interval $|a_n - b_n|$ halves every iteration i.e,$|a_n - b_n| = |b - a|2^{-n}$. Hence x_n converges to a root of $f(x)$.

Bisection method

Example: $f(x) = x^3 - x - 2$
$a = 1, b = 2;$
$f(a) = 1^3 - 1 - 2 = -2, f(b) = 2^3 - 2 - 2 = 4;$
First iteration: $x_0 = \frac{a+b}{2} = 1.5, f(x_0) = -0.125$

a_n	b_n	x_n	$f(x_n)$
1	2	1.5	-0.125
1.5	2	1.75	1.6093750
1.5	1.75	1.625	0.6660156
1.5	1.625	1.5625	0.2521973
1.5	1.5625	1.5312500	0.0591125
1.5	1.5312500	1.5156250	-0.0340538
1.5156250	1.5312500	1.5234375	0.0122504
1.5156250	1.5234375	1.5195313	-0.0109712
1.5195313	1.5234375	1.5214844	0.0006222
1.5195313	1.5214844	1.5205078	-0.0051789
1.5205078	1.5214844	1.5209961	-0.0022794
1.5209961	1.5214844	1.5212402	-0.0008289
1.5212402	1.5214844	1.5213623	-0.0001034
1.5213623	1.5214844	1.5214233	0.0002594
1.5213623	1.5214233	1.5213928	0.0000780

So, the root is approx 1.521

Bisection method

Example: $f(x) = x^3 - x^2 + x - 1$
$a = -5, b = 5$
$f(-5) = -106 < 0, f(5) = 5^3 - 5^2 + 5 - 1 = 104 > 0$

x_n	$f(x_n)$	$\|a_n - b_n\|$
0	-1	5
2.5	10.875	2.5
1.25	0.64063	1.25
0.625	-0.52148	0.625
0.9375	-0.11743	0.3125
1.0938	0.20590	0.15625
1.0156	0.03174	0.07813
0.9766	-0.04579	0.03906
0.9961	-0.00778	0.01953
1.0059	0.01179	0.00977
1.0010	0.00196	0.00488
0.9985	-0.00293	0.00244
0.9998	-0.00049	0.00122
1.0004	0.00073	0.00061
1.0001	0.00012	0.00031
0.9999	-0.00018	0.00015
1.0000	-0.00003	0.00008

So, the root is approximately 1.0000

Bisection method

Example: $f(x) = x\cos x - 2x^2 + 3x - 1$
$a = 0.2, b = 0.3$

x_n	$f(x_n)$	$\lvert a_n - b_n \rvert$
0.25	-0.13277	0.05000
0.275	-0.06158	0.02500
0.2875	-0.02711	0.01250
0.2937	-0.01016	0.00625
0.2969	-0.00176	0.00312
0.2984	0.00243	0.00156
0.2977	0.00034	0.00078
0.2973	-0.00071	0.00039
0.2975	-0.00019	0.00020
0.2976	0.00008	0.00010

So, the root is approximately 0.2976

Bisection method: Matlab code

```
% a = left end point of interval containing the root
% b = right end point of interval containing the root
% tolx = error tolerance in x
% tolf = error tolerance in the function value
% N = the current iteration number
% Nmax = maximum number of iterations
% fcn.m is the name of the file containing the function
format long
```

Bisection method : Matlab code

$a = 1;$
$b = 4;$
$N = 1;$
$Nmax = 50;$
$tolx = .001;$
$tolf = 0.000001;$
$fa = feval('fcn', a);$
$fb = feval('fcn', b);$
$m = (a + b)/2;$
$fm = feval('fcn', m);$
while $(abs(b - a) > tolx)\&(abs(fm) > tolf)\&(N < Nmax)$
$[N, a, b, m, fm]$
if $fa * fm <= 0;$
$b = m;$
$fb = fm;$
else $a = m;$
$fa = fm;$
end
$m = (a + b)/2;$
$fm = feval('fcn', m);$
$N = N + 1;$
end

False Position method

In bisection we split the interval at $\frac{a+b}{2}$ and move from (a, b) to $(a, \frac{a+b}{2})$ or $(\frac{a+b}{2}, b)$
Here we split the intervals at the point $c = \frac{af(b)-bf(a)}{f(b)-f(a)}$, the x-intercept of line joining $(a, f(a)), (b, f(b))$

Newton's method

Using the linear approximation $f(x) = f(x_0) + f'(x_0)(x - x_0)$, we get that the root x satisfies $0 = f(x_0) + f'(x_0)(x - x_0)$ if x_0 is close to the root.
Solving this we get $x = x_0 - \frac{f(x_0)}{f'(x_0)}$
We iterate this to get $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

Newton's method

If we want to find the value of $\sqrt{2}$, we need to solve for $f(x) = x^2 - 2 = 0$

The Newton iteration is $x_{n+1} = x_n - \frac{x_n^2 - 2}{2x_n} = \frac{x_n}{2} + \frac{1}{x_n}$

Secant method

How do we calculate the derivatives?
Use $f'(x_n) \approx \frac{f(x_n)-f(x_{n-1})}{x_n-x_{n-1}}$
We get the iteration,

$$x_{n+1} = x_n - \frac{f(x_n)}{\frac{f(x_n)-f(x_{n-1})}{x_n-x_{n-1}}}$$

Steffensen's method

$$h = f(x_n)$$

$$g(x_n) = \frac{f(x+h) - f(x)}{h}$$

,

$$x_{n+1} = x_n - \frac{f(x_n)}{g(x_n)}$$

FIxed Point Iteration

How do we solve for the fixed point $f(x) = x$
One idea: We can find the roots of $g(x) = f(x) - x$
Another idea: start with some value x_0, find $f(x_0), f(f(x_0)), f(f(f(x_0))) \cdots$ If the sequence converges then the limit x has to satisfy $f(x) = x$!!