A SURVEY OF WALL'S FINITENESS OBSTRUCTION

STEVE FERRY AND ANDREW RANICKI

Contents

Introduction

Wall's finiteness obstruction is an algebraic K-theory invariant which decides if a finitely dominated space is homotopy equivalent to a finite CW complex. The object of this survey is to describe the invariant and some of its many applications to the classification of manifolds. The book of Varadarajan [27] and the survey of Mislin [17] deal with the finiteness obstruction from a more homotopy theoretic point of view.

1. Finite domination

A space is finitely dominated if it is a homotopy retract of a finite complex. More formally:

Definition 1.1. A topological space X is *finitely dominated* if there exists a finite CW complex K with maps $d : K \to X$, $s : X \to K$ and a homotopy $d \circ s \simeq \mathrm{id}_X : X \to X$.

Example 1.2. (i) A compact ANR X is finitely dominated (Borsuk [2]). In fact, a finite dimensional ANR X can be embedded in $\mathbb{R}^N$ (N large), and X is a retract of an open neighbourhood $U \subset \mathbb{R}^N$ – there exist a retraction $r : U \to X$ and a compact polyhedron $K \subset U$ such that $X \subset K$, so that the restriction $d = r| : K \to X$ and the inclusion $s : X \to K$ are such that $d \circ s = \mathrm{id}_X : X \to X$.
(ii) A compact topological manifold is a compact ANR, and hence finitely dominated.

The problem of deciding if a compact ANR is homotopy equivalent to a finite CW complex was first formulated by Borsuk [3]. (The problem was solved affirmatively for manifolds by Kirby and Siebenmann in 1969, and in general by West in 1974 – see section 8 below.) The problem of deciding if a finitely dominated space is homotopy equivalent to a finite CW complex was first formulated by

J.H.C.Whitehead. Milnor [16] remarked: "It would be interesting to ask if every space which is dominated by a finite complex actually has the homotopy type of a finite complex. This is true in the simply connected case, but seems like a difficult problem in general."

Here is a useful recognition criterion for recognizing finite domination:

Proposition 1.3. *A CW complex X is finitely dominated if and only if there is a homotopy $h_t : X \to X$ such that $h_0 = id$ and $h_1(X)$ has compact closure.*

Proof. If $d : K \to X$ is a finite domination with right inverse s, let h_t be a homotopy from the identity to $d \circ s$. Since $h_1(X) \subset d(K)$, the closure of $h_1(X)$ is compact in X. Conversely, if the closure of $h_1(X)$ is compact in X, let K be a finite subcomplex of X containing $h_1(X)$. Setting d equal to the inclusion $K \to X$ and s equal to $h_1 : X \to K$ shows that X is finitely dominated. □

It is possible to relate finitely dominated spaces, finitely dominated CW complexes and spaces of the homotopy type of CW complexes, as follows.

Proposition 1.4. *(i) A finitely dominated topological space X is homotopy equivalent to a countable CW complex.*
(ii) If X is homotopy dominated by a finite k-dimensional CW complex, then X is homotopy equivalent to a countable $(k+1)$-dimensional CW complex.

Proof. The key result is the trick of Mather [15], which shows that if $d : K \to X$, $s : X \to K$ are maps such that $d \circ s \simeq \mathrm{id}_X : X \to X$ then X is homotopy equivalent to the mapping telescope of $s \circ d : K \to K$. This requires the calculus of mapping cylinder, which we now recall.
By definition, the mapping cylinder of a map $f : K \to L$ is the identification space

$$M(f) \ = \ (K \times [0,1] \cup L)/((x,1) \simeq f(x)) \ .$$

We shall use three general facts about mapping cylinders:

- If $f : K \to L$ and $g : L \to M$ are maps and $k : K \to M$ is homotopic to $g \circ f$, the mapping cylinder $M(k)$ is homotopy equivalent rel $K \cup M$ to the concatenation of the mapping cylinders $M(f)$ and $M(g)$ rel $K \cup M$.
- If $f, g : K \to L$ with $f \sim g$, then the mapping cylinder of f is homotopy equivalent to the mapping cylinder of g rel $K \cup L$.
- Every mapping cylinder is homotopy equivalent to its base rel the base.

The mapping telescope of a map $\alpha : K \to K$ is the countable union

$$\bigcup_{i=0}^{\infty} M(\alpha) \ = \ \bigcup_{i=0}^{\infty} K \times [i, i+1]/\{(x,i) \simeq (\alpha(x), i+1)\} \ .$$

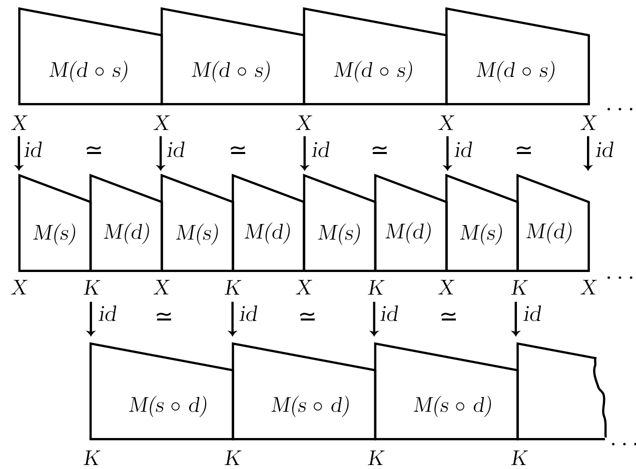
For any maps $d : K \to X$, $s : X \to K$ we have

$$\bigcup_{i=0}^{\infty} M(d \circ s) \ = \ X \times I \cup \bigcup_{i=0}^{\infty} M(s \circ d)$$

with $\bigcup_{i=0}^{\infty} M(s \circ d)$ a deformation retract, so that

$$\bigcup_{i=0}^{\infty} M(d \circ s) \ \simeq \ \bigcup_{i=0}^{\infty} M(s \circ d) \ .$$

To see why this holds, note that $\bigcup_{i=0}^{\infty} M(d \circ s)$ is homotopy equivalent to an infinite concatenation of alternating $M(d)$'s and $M(s)$'s which can also be thought of as an infinite concatenation of $M(s)$'s and $M(d)$'s. Essentially, we're reassociating an infinite product. Here is a picture of this part of the construction.



(i) If $d : K \to X$, $s : X \to K$ are such that $d \circ s \simeq \mathrm{id}_X : X \to X$ there is defined a homotopy idempotent of a finite CW complex

$$\alpha \;=\; s \circ d : K \to K \;,$$

with $\alpha \circ \alpha \simeq \alpha : K \to K$. We have homotopy equivalences

$$X \simeq X \times [0,\infty) \simeq \bigcup_{i=0}^{\infty} M(\mathrm{id}_X) \simeq \bigcup_{i=0}^{\infty} M(d \circ s) \simeq \bigcup_{i=0}^{\infty} M(s \circ d) = \bigcup_{i=0}^{\infty} M(\alpha).$$

The mapping telescope $\bigcup_{i=0}^{\infty} M(\alpha)$ is a countable CW complex.
(ii) As for (i), but with K k-dimensional. □

This proposition is comforting because it shows that the finiteness problem for arbitrary topological spaces reduces to the finiteness problem for CW complexes. One useful consequence of this is that we can use the usual machinery of algebraic topology, including the Hurewicz and Whitehead theorems, to detect homotopy equivalences.

Proposition 1.5. *(Mather [15]) A topological space X is finitely dominated if and only if $X \times S^1$ is homotopy equivalent to a finite CW complex.*

Proof. The mapping torus of a map $\alpha : K \to K$ is defined (as usual) by

$$T(\alpha) \;=\; (K \times [0,1])/\{(x,0) \simeq (\alpha(x),1)\} \;.$$

For any maps $d : K \to X$, $s : X \to K$ there is defined a homotopy equivalence

$$T(d \circ s : X \to X) \to T(s \circ d : K \to K) \;;\; (x,t) \mapsto (s(x),t) \;.$$

If $d \circ s \simeq \mathrm{id}_X : X \to X$ and K is a finite CW complex we thus have homotopy equivalences

$$X \times S^1 \simeq T(\mathrm{id}_X) \simeq T(s \circ d)$$

with $T(s \circ d)$ a finite CW complex.
Conversely, if $X \times S^1$ is homotopy equivalent to a finite CW complex K then the maps

$$d : K \simeq X \times S^1 \xrightarrow{proj.} X ,$$

$$s : X \xrightarrow{incl.} X \times S^1 \simeq K$$

are such that $d \circ s \simeq \mathrm{id}_X$, and X is dominated by K. □

2. The projective class group K_0

Let Λ be a ring.

Definition 2.1. A Λ-module P is *finitely generated projective* if it is a direct summand of a finitely generated free Λ-module Λ^n, with $P \oplus Q = \Lambda^n$ for some direct complement Q.

A Λ-module P is finitely generated projective if and only if P is isomorphic to $\mathrm{im}(p)$ for some projection $p = p^2 : \Lambda^n \to \Lambda^n$.

Definition 2.2. The *projective class group* $K_0(\Lambda)$ is the Grothendieck group of stable isomorphism classes of finitely generated projective Λ-modules.
The *reduced projective class group* $\widetilde{K}_0(\Lambda)$ is the quotient of $K_0(\Lambda)$ by the subgroup generated by formal differences $[\Lambda^m] - [\Lambda^n]$ of finitely generated free modules.

Thus an element of $\widetilde{K}_0(\Lambda)$ is an equivalence class $[P]$ of finitely generated projective Λ-modules, with $[P_1] = [P_2]$ if and only if there are finitely generated free Λ-modules F_1 and F_2 so that $P_1 \oplus F_1$ is isomorphic to $P_2 \oplus F_2$. In particular, $[P]$ is trivial if and only if P is *stably free*, that is, if there is a finitely generated free module F so that $P \oplus F$ is free.

Example 2.3. There are many groups π for which

$$\widetilde{K}_0(\mathbb{Z}[\pi]) \ = \ 0 \ ,$$

including virtually polycyclic groups, a class which includes free and free abelian groups.

At present, no example is known of a torsion-free infinite group π with $\widetilde{K}_0(\mathbb{Z}[\pi]) \neq 0$. Indeed, Hsiang has conjectured that $\widetilde{K}_0(\mathbb{Z}[\pi]) = 0$ for any torsion-free group π. On the other hand:

Example 2.4. (i) There are many finite groups π for which

$$\widetilde{K}_0(\mathbb{Z}[\pi]) \ \neq \ 0 \ ,$$

including the cyclic group $\mathbb{Z}_{23}$.
(ii) The reduced projective class group of the quaternion group

$$Q(8) \ = \ \{\pm 1, \pm i, \pm j, \pm k\}$$

is

$$\widetilde{K}_0(\mathbb{Z}[Q(8)]) \ = \ \mathbb{Z}_2 \ ,$$

generated by the finitely generated projective $\mathbb{Z}[Q(8)]$-module

$$P \ = \ \mathrm{im}\bigg(\begin{pmatrix} 1-8N & 21N \\ -3N & 8N \end{pmatrix} \ : \ \mathbb{Z}[Q(8)] \oplus \mathbb{Z}[Q(8)] \to \mathbb{Z}[Q(8)] \oplus \mathbb{Z}[Q(8)] \bigg)$$

with $N = \sum\limits_{g \in Q(8)} g$.

We refer to Oliver [18] for a survey of the computations of $\widetilde{K}_0(\mathbb{Z}[\pi])$ for finite groups π.

3. The finiteness obstruction

Here is the statement of Wall's theorem.

Theorem 3.1. *([28]) (i) A finitely dominated space X has a finiteness obstruction*

$$\sigma(X) \in \widetilde{K}_0(\mathbb{Z}[\pi_1(X)])$$

such that $\sigma(X) = 0$ if and only if X is homotopy equivalent to a finite CW complex.
(ii) If π is a finitely presented group then every element $\sigma \in \widetilde{K}_0(\mathbb{Z}[\pi])$ is the finiteness obstruction of a finitely dominated CW complex X with $\sigma(X) = \sigma$, $\pi_1(X) = \pi$.

Outline of proof (i) Here is an extremely condensed sketch of Wall's argument from [28]. If $d : K \to X$ is a finite domination with X a CW complex, we can assume that d is an inclusion by replacing X, if necessary, by the mapping cylinder of d. For each $\ell \geq 2$, we then have a split short exact sequence of abelian groups

$$0 \to \pi_{\ell+1}(X, K) \to \pi_\ell(K) \to \pi_\ell(X) \to 0 \ .$$

Wall gives a special argument to show that d can be taken to induce an isomorphism on π_1 and then shows that $\pi_{\ell+1}(X, K)$ is finitely generated as a module over $\mathbb{Z}[\pi_1(X)]$, provided that $\pi_q(X, K) = 0$ for $q \leq \ell$, $\ell \geq 2$. This allows him to attach $\ell + 1$-cells to form a complex $\overline{K} \supset K$ and a map $\overline{d} : \overline{K} \to X$ extending d so that $\overline{d}$ induces isomorphisms on homotopy groups through dimension ℓ. Since $\overline{d}$ is a domination with the same right inverse s, this process can be repeated. In case $\ell \geq \dim(K)$, Wall shows that $\pi_{\ell+1}(X, K)$ is a finitely generated *projective* module over $\mathbb{Z}[\pi_1(X)]$. If $\pi_{\ell+1}(X, K)$ is free (or even stably free) we can attach $\ell + 1$-cells to kill $\pi_{\ell+1}(X, K)$ without creating new problems in higher dimensions. The result is that $\overline{d}$ is a homotopy equivalence from $\overline{K}$ to X. If this module is not stably free, we are stuck and the finiteness obstruction is defined to be

$$\sigma(X) \ = \ (-1)^{\ell+1}[\pi_{\ell+1}(X, K)] \in \widetilde{K}_0(\mathbb{Z}[\pi_1(X)]) \ .$$

(ii) Given a finite CW complex K and a nontrivial $\sigma \in \widetilde{K}_0(\mathbb{Z}[\pi_1(K)])$, here is one way to construct a CW complex with finiteness obstruction $\pm\sigma$: let σ be represented by a finitely generated projective module P and let $F = P \oplus Q$ be free of rank n. Let A be the matrix of the projection $p : F \to P \to F$ with respect to a standard basis for F. Now let

$$L = K \vee \bigvee_{i=1}^{n} S^\ell_i.$$

There is a split short exact sequence

$$0 \longrightarrow \pi_\ell(K) \underset{i_*}{\overset{r_*}{\rightleftarrows}} \pi_\ell(L) \longrightarrow \pi_\ell(L, K) \longrightarrow 0,$$

where $r : L \to K$ is the retraction which sends the S^ℓ's to the basepoint. Since

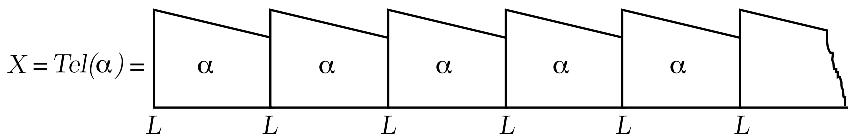
$$\pi_\ell(L, K) \cong \pi_\ell(\tilde{L}, \tilde{K}) \cong H_\ell(\tilde{L}, \tilde{K}) \cong F,$$

we can define $\alpha : L \to L$ so that $\alpha|K = id$ and so that $\alpha_* : \pi_\ell(L) \to \pi_\ell(L)$ has the matrix

$$\begin{pmatrix} id & 0 \\ 0 & A \end{pmatrix}$$

with respect to the direct sum decomposition $\pi_\ell(L) \cong \pi_\ell(K) \oplus F$. Since $A^2 = A$, it is easy to check that α is *homotopy idempotent*, i.e. that $\alpha \circ \alpha \sim \alpha$ rel K.

Let X be the infinite direct mapping telescope of α pictured below.



Let $d : L \to X$ be the inclusion of L into the top level of the leftmost mapping cylinder of X and define $s' : X \to L$ by setting s' equal to α on each copy of L and using the homotopies $\alpha \circ \alpha \sim \alpha$ to extend over the rest of X. One sees easily that $d \circ s'$ induces the identity on the homotopy groups of X and is therefore a homotopy equivalence. If ϕ is a homotopy inverse for $d \circ s'$, we have $d \circ s \sim id$, where $s = s' \circ \phi$. This means the d is a finite domination with right inverse s. It turns out that $\sigma(X) = (-1)^{\ell+1}[P]$. □

In particular, if π is a finitely presented group such that $\widetilde{K}_0(\mathbb{Z}[\pi]) \neq 0$ then there exists a finitely dominated CW complex X with $\pi_1(X) = \pi$ and such that X is not homotopy equivalent to a finite CW complex. See Ferry [8] for the construction of finitely dominated compact metric spaces (which are not ANR's, still less CW complexes) which are not homotopy equivalent to a finite CW complex.

Wall [29] obtained the finiteness obstruction of a finitely dominated CW complex from the cellular chain complex $C_*(\widetilde{X})$ of the universal cover $\widetilde{X}$ of X, proving that $C_*(\widetilde{X})$ is chain homotopy equivalent to a finite chain complex of finitely generated projective modules over $\mathbb{Z}[\pi_1(X)]$

$$\mathcal{P} : \cdots \to 0 \to P_n \xrightarrow{\partial} P_{n-1} \xrightarrow{\partial} \cdots \xrightarrow{\partial} P_1 \xrightarrow{\partial} P_0 \ .$$

Definition 3.2. The *projective class* of X is the projective class of $\mathcal{P}$

$$[X] \ = \ \sum_{i=0}^{\infty} (-1)^i [P_i] \in K_0(\mathbb{Z}[\pi_1(X)]) \ .$$

The projective class is a well-defined chain-homotopy invariant of $C_*(\widetilde{X})$, with components

$$[X] \ = \ (\chi(X), \sigma(X)) \in K_0(\mathbb{Z}[\pi_1(X)]) \ = \ K_0(\mathbb{Z}) \oplus \widetilde{K}_0(\mathbb{Z}[\pi_1(X)]) \ ,$$

with

$$\chi(X) \ = \ \sum_{i=0}^{\infty} (-1)^i \# \text{ of } i\text{-cells} \in K_0(\mathbb{Z}) \ = \ \mathbb{Z}$$

the Euler characteristic of X, and $\sigma(X)$ the finiteness obstruction.

The *instant finiteness obstruction* (Ranicki [19]) of a finitely dominated CW complex X is a finitely generated projective $\mathbb{Z}[\pi_1(X)]$-module P representing the finiteness obstruction

$$\sigma(X) \ = \ [P] \in \widetilde{K}_0(\mathbb{Z}[\pi_1(X)])$$

which is obtained directly from a finite domination $d : K \to X$, $s : X \to K$, a homotopy $h : d \circ s \simeq \mathrm{id}_X : X \to X$ and the cellular $\mathbb{Z}[\pi_1(X)]$-module chain complex $C(\widetilde{X})$ of the universal cover $\widetilde{X}$ of X, namely

$$P \ = \ \mathrm{im}(p : \mathbb{Z}[\pi_1(X)]^n \to \mathbb{Z}[\pi_1(X)]^n)$$

with

$$p \ = \ \begin{pmatrix} s \circ d & -\partial & 0 & \dots \\ -s \circ h \circ d & 1 - s \circ d & \partial & \dots \\ -s \circ h^2 \circ d & s \circ h \circ d & s \circ d & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} : \mathbb{Z}[\pi_1(X)]^n \ = \ \sum_{i=0}^{\infty} C(\widetilde{X})_i \to \sum_{i=0}^{\infty} C(\widetilde{X})_i$$

a projection of a finitely generated free $\mathbb{Z}[\pi_1(X)]$-module of rank $n = \sum\limits_{i=0}^{\infty} \#$ of i-cells.

The finiteness obstruction has many of the usual properties of the Euler characteristic χ. For instance, if X is the union of finitely dominated complexes X_1 and X_2 along a common finitely dominated subcomplex X_0, then

$$\sigma(X) = i_{1*}\sigma(X_1) + i_{2*}\sigma(X_2) - i_{0*}\sigma(X_0).$$

This is the *sum theorem* for finiteness obstructions, which was originally proven in Siebenmann's thesis [25].

The projective class of the product $X \times Y$ of finitely dominated CW complexes X, Y is given by

$$[X \times Y] \ = \ [X] \otimes [Y] \in K_0(\mathbb{Z}[\pi_1(X \times Y)]) \ ,$$

leading to the *product formula* of Gersten [10] for the finiteness obstruction

$$\sigma(X \times Y) \ = \ \chi(X) \otimes \sigma(Y) + \sigma(X) \otimes \chi(Y) + \sigma(X) \otimes \sigma(Y) \in \widetilde{K}_0(\mathbb{Z}[\pi_1(X \times Y)]) \ .$$

In particular, $\sigma(X \times S^1) = 0$, giving an algebraic proof of the result (1.5) that $X \times S^1$ is homotopy equivalent to a finite CW complex.

4. The topological space-form problem

Another problem in which a finiteness obstruction arises is the *topological space-form problem*. This is the problem of determining which groups can act freely and properly discontinuously on S^n for some n.

Swan, [26], solved a homotopy version of this problem by proving that a finite group G of order n which has periodic cohomology of period q acts freely on a finite complex of dimension $dq - 1$ which is homotopy equivalent to a $(dq-1)$-sphere. Here, d is the greatest common divisor of n and $\phi(n)$, ϕ being Euler's ϕ-function.

One might ask whether such a G can act on S^{q-1}, but this refinement leads to a finiteness obstruction. It follows from Swan's argument that G acts freely on a countable $q-1$-dimensional complex X homotopy equivalent to S^{q-1} and that X/G is finitely dominated. The finiteness obstruction of X/G need not be zero, however, so not every group with cohomology of period q can act freely on a *finite* complex homotopy equivalent to S^{q-1}. Algebraically, the point is that finite groups with q-periodic cohomology have q-periodic resolutions by finitely generated projective modules but need not have q-periodic resolutions by finitely generated free modules.

After a great deal of work involving both the finiteness obstruction and surgery theory, see Madsen, Thomas and Wall [14], it turned out that a group G acts freely on S^n for some n if and only if all of its subgroups of order p^2 and $2p$ are cyclic (the condition of Milnor). This is in contrast to the linear case. A group G acts linearly on S^n for some n if and only if all subgroups of order pq, p and q not necessarily distinct primes, are cyclic. See Davis and Milgram [6] for a book-length treatment, and Weinberger [30], p. 110, for a brief discussion.

5. The Siebenmann end obstruction

The most significant application of the finiteness obstruction to the topology of manifolds is via the end obstruction.

An end ϵ of an open n-dimensional manifold W is *tame* if there exists a sequence $W \supset U_1 \supset U_2 \supset \dots$ of finitely dominated neighbourhoods of ϵ with

$$\bigcap_i U_i \;=\; \emptyset \;,\; \pi_1(U_1) \cong \pi_1(U_2) \cong \cdots \cong \pi_1(\epsilon) \;.$$

The end is *collared* if there exists a neighbourhood of the type $M \times [0, \infty)$ for some closed $(n-1)$-dimensional manifold M, i.e. if ϵ is the interior of a compactification $W \cup M$ with boundary component M.

Theorem 5.1. *(Siebenmann* [25]*) A tame end ϵ of an open n-dimensional manifold W has an end obstruction*

$$\sigma(\epsilon) \;=\; \varinjlim_i \sigma(U_i) \in \widetilde{K}_0(\mathbb{Z}[\pi_1(\epsilon)])$$

such that $\sigma(\epsilon) = 0$ if (and for $n \geq 6$ only if) ϵ can be collared.

Novikov's 1965 proof of the topological invariance of the rational Pontrjagin classes made use of the end obstruction in the unobstructed case when π is a free abelian group. The subsequent work of Lashof, Rothenberg, Casson, Sullivan et. al. ([23]) on the Hauptvermutung for high-dimensional manifolds made overt use of the end obstruction.

See sections 8 and 9 below for a brief account of the applications of the end obstruction to splitting theorems and triangulation of high-dimensional manifolds.

See Hughes and Ranicki [11] for a book-length treatment of ends and the end obstruction.

6. Connections with Whitehead torsion

The finiteness obstruction deals with the existence of a finite CW complex K in a homotopy type, while Whitehead torsion deals with the uniqueness of K. There are many deep connections between the finiteness obstruction and Whitehead torsion, which on the purely algebraic level correspond to the connections between the algebraic K-groups K_0, K_1 (or rather $\widetilde{K}_0$, Wh)

The splitting theorem of Bass, Heller and Swan [1]

$$Wh(\pi \times \mathbb{Z}) \;=\; Wh(\pi) \oplus \widetilde{K}_0(\mathbb{Z}[\pi]) \oplus \widetilde{Nil}_0(\mathbb{Z}[\pi]) \oplus \widetilde{Nil}_0(\mathbb{Z}[\pi])$$

involves a split injection

$$\widetilde{K}_0(\mathbb{Z}[\pi]) \to Wh(\pi \times \mathbb{Z}) \;;\; [P] \to \tau(z : P[z, z^{-1}] \to P[z, z^{-1}]) \;.$$

If X is a finitely dominated space then 1.5 gives a homotopy equivalence $\phi : X \times S^1 \to K$ to a finite CW complex K, uniquely up to simple homotopy equivalence. Ferry [9] identified the finiteness obstruction $\sigma(X) \in \widetilde{K}_0(\mathbb{Z}[\pi_1(X)])$ with the Whitehead torsion $\tau(f) \in Wh(\pi \times \mathbb{Z})$ of the composite self homotopy equivalence of a finite CW complex

$$f \;:\; K \xrightarrow{\phi^{-1}} X \times S^1 \xrightarrow{\mathrm{id}_X \times -1} X \times S^1 \xrightarrow{\phi} K \;.$$

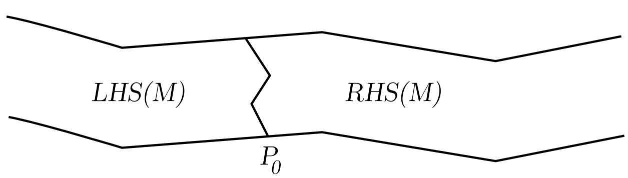
In Ranicki [20],[21] it was shown that $\sigma(X) \mapsto \tau(\phi)$ corresponds to the split injection

$$\widetilde{K}_0(\mathbb{Z}[\pi]) \to Wh(\pi \times \mathbb{Z}) \;;\; [P] \to \tau(-z : P[z, z^{-1}] \to P[z, z^{-1}])$$

which is different from the original split injection of [1].

7. The splitting obstruction

The finiteness obstruction arises in most classification problems in high-dimensional topology. Loosely speaking, proving that two manifolds are homeomorphic involves decomposing them into homeomorphic pieces. Finiteness obstructions arise as obstructions to splitting a manifold into pieces. The nonsimply-connected version of Browder's $M \times \mathbb{R}$ Theorem is a case in point. In [4], Browder proved that if M^n, $n \geq 6$, is a PL manifold without boundary, $f : M \to K \times \mathbb{R}^1$ is a (PL) proper homotopy equivalence, and K is a simply-connected finite complex, then M is homeomorphic to $N \times \mathbb{R}^1$ for some closed manifold N homotopy equivalent to K.



In case K is connected but not simply-connected, a finiteness obstruction arises. Here is a quick sketch of the argument: It is not difficult to show that M is 2-ended. The proper homotopy equivalence $f : M \to K \times \mathbb{R}^1$ gives us a proper PL map $p : M \to \mathbb{R}$. If $c \in \mathbb{R}$ is not the image of any vertex, then $p^{-1}(c)$ is a bicollared PL submanifold of M which separates the ends. Connected summing components along arcs allows us to assume that $P_0 = p^{-1}(c)$ is connected and a disk-trading argument similar to one in Browder's paper allows us to assume that $\pi_1 P_0 \to \pi_1 M$ is an isomorphism. See Siebenmann [25] for details. An application of the recognition criterion discussed in the third paragraph of this paper shows that the two components of $M - P_0$, which we denote by $RHS(M)$ and $LHS(M)$, respectively, are finitely dominated. By the sum theorem,

$$\sigma(RHS(M)) + \sigma(LHS(M)) = 0.$$

It turns out that the vanishing of $\sigma(RHS(M)) = -\sigma(LHS(M))$ is necessary and sufficient for M to be homeomorphic to a product $N\times\mathbb{R}$, provided that $\dim(M) \geq 6$. This is one of the main results of [25].

It is possible to realize the finiteness obstruction σ on an n-dimensional manifold M^n proper homotopy equivalent to $K \times \mathbb{R}$ for some finite K whenever $\sigma + (-1)^{n-1}\sigma^* = 0$ and $n \geq 6$. If we only require that M be properly dominated by some $K\times\mathbb{R}$, then any finiteness obstruction σ can be realized. A similar obstruction arises in the problem of determining whether a map $p : M^n \to S^1$ is homotopic to the projection map of a fiber bundle (Farrell [7]).

The geometric splitting of two-ended open manifolds into right and left sides is closely related to the proof of the algebraic splitting theorem of Bass, Heller and Swan [1] for $Wh(\pi \times \mathbb{Z})$ – see Ranicki [22].

8. The triangulation of manifolds

The finiteness obstruction arises in connection with another of the fundamental problems of topology: *Is every compact topological manifold without boundary homeomorphic to a finite polyhedron?* We will examine this problem in much greater detail.

The triangulation problem was solved affirmatively for two-dimensional manifolds by Rado in 1924 and for three-dimensional manifolds by Moise in 1952. Higher dimensions proved less tractable,[1] a circumstance which encouraged the formulation of weaker questions such as the following *homotopy-triangulation problem*: *Does every compact topological manifold have the homotopy type of some finite polyhedron?*

The first solution of this problem came as a corollary to Kirby and Siebenmann's theory of PL triangulations of high-dimensional topological manifolds. By a theorem of Hirsch, every topological manifold M^n has a well-defined stable topological normal disk bundle. The total space of this bundle is a closed neighborhood of M in some high-dimensional euclidean space. In [12], Kirby and Siebenmann proved that a topological n-manifold, $n \geq 6$, has a PL structure if and only if this stable normal bundle reduces from TOP to PL. As an immediate corollary, they deduced that every compact topological manifold has the homotopy type of a finite polyhedron, since each M is homotopy equivalent to the total space of the unit disk bundle of its normal disk bundle and the total space of the normal disk bundle is a PL manifold because its normal bundle is trivial. The argument of Kirby and Siebenmann also shows that each compact topological manifold has a well-defined simple homotopy type. A more refined argument, see p.104 of Kirby and Siebenmann [13], shows that every closed topological manifold of dimension ≥ 6 is a TOP handlebody. From this it follows immediately that every compact topological manifold is homotopy equivalent to a finite CW complex and therefore to a finite polyhedron.

This positive solution to the homotopy-triangulation problem suggests that we should look for large naturally-occurring classes of compact topological spaces which

[1] In fact, Casson has shown that there are compact four-manifolds without boundary which are not homeomorphic to finite polyhedra. The question is still open in dimensions greater than or equal to five.

have the homotopy types of finite polyhedra. In 1954, K. Borsuk [3] conjectured that every compact metrizable ANR should have the homotopy type of a finite polyhedron. This became widely known as *Borsuk's Conjecture*.

The Borsuk Conjecture was solved by J. E. West, [31], using results of T. A. Chapman, which, in turn, were based on an infinite-dimensional version of Kirby-Siebenmann's handle-straightening argument. In a nutshell, Chapman proved that every compact manifold modeled on the Hilbert cube ($\equiv \prod_{i=1}^{\infty}[0,1]$) is homotopy equivalent to a finite complex and West showed that every compact ANR[2] is homotopy equivalent to a compact manifold modeled on the Hilbert cube. A rather short finite-dimensional proof of the topological invariance of Whitehead torsion, together with the Borsuk Conjecture was given by Chapman in [5]. See Ranicki and Yamasaki [24] for a more recent proof, which makes use of controlled algebraic K-theory.

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[2] A compact metrizable space X is an ANR if and only if it embeds as a neighborhood retract in separable Hilbert space. If X has finite covering dimension $\leq n$, separable Hilbert space can be replaced by $\mathbb{R}^{2n+1}$.

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S.F.: Dept. of Mathematics, Binghamton University, Binghamton, NY 13901, USA

A.R.: Dept. of Mathematics and Statistics, University of Edinburgh, Edinburgh EH9 3JZ, Scotland, UK