MA 491 Problem set #9

Power series

1. Let \( p \) and \( q \) be real numbers with \( 1/p - 1/q = 1, \) \( 0 < p \leq 1/2. \) Show that
\[
p + \frac{1}{2}p^2 + \frac{1}{3}p^3 + \ldots = q - \frac{1}{2}q^2 + \frac{1}{3}q^3 - \ldots
\]

2. Find the power series expansion for
\[
\frac{1}{(x^2 + 5x + 6)}.
\]
(Hint: Partial fractions.)

3. Prove that the value of the \( n \)th derivative of \( x^3/(x^2 - 1) \) for \( x = 0 \) is zero if \( n \) is even and \(-n!\) if \( n \) is odd and greater than 1.

4. Show that the functional equation
\[
f \left( \frac{2x}{1 + x^2} \right) = (1 + x^2)f(x)
\]
is satisfied by
\[
f(x) = 1 + \frac{1}{3}x^2 + \frac{1}{5}x^4 + \frac{1}{7}x^6 + \ldots, \quad |x| < 1.
\]

Easy Putnam Problems

5. Let \( \{x\} \) denote the distance between the real number \( x \) and the nearest integer. For each positive integer \( n, \) evaluate
\[
S_n = \sum_{m=1}^{6n-1} \min \left( \left\{ \frac{m}{6n} \right\}, \left\{ \frac{m}{3n} \right\} \right).
\]
(Here \( \min(a,b) \) denotes the minimum of \( a \) and \( b. \))

6. A right circular cone has base of radius 1 and height 3. A cube is inscribed in the cone so that one face of the cube is contained in the base of the cone. What is the side-length of the cube?

7. Let \( s \) be any arc of the unit circle lying entirely in the first quadrant. Let \( A \) be the area of the region lying below \( s \) and above the \( x \)-axis and let \( B \) be the area of the region lying to the right of the \( y \)-axis and to the left of \( s \). Prove that \( A + B \) depends only on the arc length and not on the position of \( s \).

Harder Putnam Problems

8. Let \( f \) be a twice-differentiable real-valued function satisfying
\[
f(x) + f''(x) = -xg(x)f'(x),
\]
where \( g(x) \geq 0 \) for all real \( x. \) Prove that \( |f(x)| \) is bounded.
9. Let $a_{m,n}$ denote the coefficient of $x^n$ in the expansion of $(1 + x + x^2)^m$. Prove that for all integers $k \geq 0$,

$$0 \leq \sum_{i=0}^{\left\lfloor \frac{k}{4} \right\rfloor} (-1)^i a_{k-i,i} \leq 1.$$ 

10. Let $f$ be a function on the real line with continuous third derivative. Prove that there exists a point such that

$$f(a) \cdot f'(a) \cdot f''(a) \cdot f'''(a) \geq 0.$$