GCD Problems

1. Find all functions \( f \) which satisfy the three conditions
   (1) \( f(x,x)=x \)
   (2) \( f(x,y)=f(y,x) \)
   (3) \( f(x,y)=f(x,x+y) \)
assuming that the variables and the values of \( f \) are positive integers.

2. Prove that the fraction \( \frac{21n+4}{14n+3} \) is irreducible for every natural number \( n \).

3. Find the smallest positive integer \( a \) for which
   \[ 1001x + 770y = 1000000 + a \]
is possible and show that it then has 100 solutions in positive integers.

4. A man goes to a stream with a 9-pint container and a 16-pint container. What should he do to get one pint of water in the 16-pint container?

Easy Putnam Problems

5. Find all real-valued continuously differentiable functions \( f \) on the real line such that for all \( x \)
   \[ (f(x))^2 = \int_0^x ((f(t))^2 + (f'(t))^2) \, dt + 1990 \]

6. Let \( S \) be a set of \( 2 \times 2 \) integer matrices whose entries \( a_{ij} \) (1) are all squares of integers, and, (2) satisfy \( a_{ij} \leq 200 \). Show that if \( S \) has more than 50387 (= \( 15^4 - 15^2 - 15 + 2 \) elements, then it has two elements that commute.

7. A \( 2 \times 3 \) rectangle has vertices at \((0,0), (2,0), (0,3), \) and \((2,3)\). It rotates \( 90^\circ \) clockwise about the point \((2,0)\). It then rotates \( 90^\circ \) clockwise about the point \((5,0)\), then \( 90^\circ \) clockwise about the point \((7,0)\), and finally \( 90^\circ \) clockwise about the point \((10,0)\). (The side originally on the \( x \)-axis is now back on the \( x \)-axis.) Find the area of the region above the \( x \)-axis and below the curve traced out by the point whose initial position is \((1,1)\).

8. Let \( A \) and \( B \) be different \( n \times n \) matrices with real entries. If \( A^3 = B^3 \) and \( A^2 B = B^2 A \), can \( A^2 + B^2 \) be invertible?

Harder Putnam Problems

9. Prove that if \( \sum_{n=1}^{\infty} a_n \) is a convergent series, then so is \( \sum_{n=1}^{\infty} (a_n)^{n/(n+1)} \).

10. If every point in the plane is painted one of three colors, do there necessarily exist two points of the same color exactly one inch apart? What if “three” is replaced by “nine”?