MA 491 Problem set #2

Pigeonhole problems

1. Let $A$ be any set of 20 distinct integers chosen from the arithmetic progression $1, 4, 7, \ldots, 100$. Prove that there must be two distinct integers in $A$ whose sum is 104.

2. Show that if there are $n$ people at a party, then two of them know the same number of people (among those present).

3. Fifteen chairs are evenly placed around a circular table on which are name cards for fifteen guests. The guests fail to notice these cards until they have sat down, and it turns out that no one is sitting in front of his own card. Prove that the table can be rotated so that at least two of the guests are simultaneously correctly seated.

4. Prove that no seven positive integers, not exceeding 24, can have sums of all subsets different.

Putnam problems

5. A composite (positive integer) is a product $ab$ with $a$ and $b$ not necessarily distinct integers in $\{2, 3, 4, \ldots\}$. Show that every composite is expressible as $xy+xz+yz+1$ with $x, y,$ and $z$ positive integers.

6. Evaluate $\int_0^a \int_0^b e^\max\{b^2x^2,a^2y^2\} \, dy \, dx$, where $a$ and $b$ are positive.

7. A dart, thrown at random, hits a square target. Assuming that any two parts of the target of equal area are equally likely to be hit, find the probability that the point hit is nearer to the center than to any edge. Express your answer in the form $(a\sqrt{b}+c)/d$, where $a, b, c, d$ are positive integers.

8. Let

$$T_0 = 2, \ T_1 = 3, \ T_2 = 6$$

and for $n \geq 3$,

$$T_n = (n+4)T_{n-1} - 4nT_{n-2} + (4n-8)T_{n-3}$$

The first few terms are

$$2, \ 3, \ 6, \ 14, \ 152, \ 784, \ 5168, \ 40576, \ 363392.$$ 

Find, with proof, a formula for $T_n$ of the form $T_n = A_n + B_n$, where $\{A_n\}$ and $\{B_n\}$ are well-known sequences.

Harder Putnam problems

9. Is $\sqrt{2}$ the limit of a sequence of numbers of the form $\sqrt{n} - \sqrt{m}$, $(n, m = 0, 1, 2, \ldots)$?

10. Consider a paper punch that can be centered at any point of the plane and that, when operated, removes from the plane precisely those points whose distance from the center is irrational. How many punches are needed to remove every point?