1. (Larson, 5.4.24) Use the technique of generating functions to solve the recurrence relation
\[ a_0 = 1, a_1 = 0, a_2 = -5 \text{ and for } n > 3, \]
\[ a_n = 4a_{n-1} - 5a_{n-2} + 2a_{n-3}. \]

2. (Larson, 5.4.25) Use the technique of generating functions to show that the \( n \)th Fibonacci number \( F_n \) is equal to:
\[ F_n = \frac{(1 + \sqrt{5})/2)^n - ((1 - \sqrt{5})/2)^n}{\sqrt{5}}. \]

3. (Larson, 5.4.29) Show that \( \sum_{n=0}^{\infty} \frac{\sin(n\theta)}{n!} = \sin(\sin(\theta))e^{\cos(\theta)}. \)

4. (Larson, 5.4.23) Let \( T_n = \sum_{i=1}^{n} (-1)^{i+1}/(2i - 1), \) and \( T = \lim_{n \to \infty} T_n. \) Show that:
\[ \sum_{n=1}^{\infty} T_n - T = \frac{\pi^2}{8} - \frac{1}{4}. \]

**Easier Putnam Problems**

5. For \( 0 < x < 1, \) express \( \sum_{n=0}^{\infty} \frac{x^{2n}}{1-x^{2n}} \) as a rational function of \( x \)

6. Define \( S_0 = 1. \) For \( n \geq 1, \) let \( S_n \) be the number of \( n \times n \) symmetric matrices with nonnegative entries, and all row sums equal to 1. Prove:
\[ \begin{align*}
(a) & \quad S_{n+1} = S_n + nS_{n-1} \\
(b) & \quad \sum_{n=0}^{\infty} S_n x^n/n! = e^{x+x^2/2}.
\end{align*} \]

7. (Larson, 5.4.20, taken from Putnam Exam) If \( n \) is a positive integer, let \( (B(n)) \) be the number of ones in the base 2 expression for \( n. \) For example, \( B(6) = 2, \) \( B(15) = 4. \) Determine whether or not \( e^{\sum_{n=1}^{\infty} B(n)/n(n+1)} \) is a rational number.

**Harder Putnam Problems**

8. For each positive integer \( n, \) let \( f_n(x) \) denote the function:
\[ f_n(x) = \sum_{0 \leq k \leq n/2} \frac{\binom{n}{2k} x^k}{\sum_{0 \leq k \leq n/2} \binom{n}{2k+1} x^k}. \]
Express \( f_{n+1}(x) \) rationally in terms of \( f_n(x) \) and \( x. \) Determine \( \lim_{n \to \infty} f_n(x) \) for all of the values of \( x \) that you can.

9. Let \( f_0(x) = e^x \) and \( f_{n+1}(x) = xf_n'(x) \) for \( n = 0, 1, 2, \ldots. \) Show that
\[ \sum_{n=0}^{\infty} \frac{f_n(1)}{n!} = e^e. \]