Problem 1. Use Lagrange multipliers to find the maximum and minimum values of \( f(x, y) = x + y^2 \) subject to the constraint \( g(x, y) = x^2 + 2y^2 = 4 \).

Solution: \( \nabla f = \langle 1, 2y \rangle \) and \( \nabla g = \langle 2x, 4y \rangle \). \( \begin{vmatrix} 1 & 2y \\ 2x & 4y \end{vmatrix} = 4y - 4xy = 0 \). So \( 4y(1 - x) = 0 \) and \( y = 0 \) or \( x = 1 \). If \( y = 0 \), substituting into \( x^2 + 2y^2 = 4 \) gives \( x = \pm 2 \) and \( f = \pm 2 \). If \( x = 1 \), \( 2y^2 = 3 \), so \( y = \pm \sqrt{3/2} \) and \( f = 5/2 \). Thus, the minimum is \(-2\) at \((-2, 0)\) and the maximum is \(5/2\) at \((1, \pm \sqrt{3/2})\).

Problem 2. Evaluate the iterated integral \( \int_0^1 \int_0^v \sqrt{1 - v^2} \, du \, dv \).

Solution: \( \int_0^v \sqrt{1 - v^2} \, du = \int_0^v v \sqrt{1 - v^2} \, dv \). Let \( w = 1 - v^2 \). Then \( dw = -2v \, dv \) and \( \int_0^1 v \sqrt{1 - v^2} \, dv = -\frac{1}{2} \int_1^0 \sqrt{w} \, dw = \frac{1}{3} \). Don’t forget to change the u-limits to w-limits!

Problem 3. Change the integral \( \int_0^a \int_0^{\sqrt{a^2 - y^2}} (x^2 + y^2)^{3/2} \, dx \, dy \) to polar coordinates. \textbf{Do not evaluate the integral}.

Solution: The region of integration is the part of \( x^2 + y^2 = a^2 \) in the first quadrant, so the integral is \( \int_0^a \int_0^r r^3 \, r \, dr \, d\theta \).

Problem 4. Evaluate the integral \( \int_0^3 \int_{x^2}^9 y \cos(x^2) \, dx \, dy \) by reversing the order of integration.

Solution: The reversed integral is \( \int_0^9 \int_0^{\sqrt{x}} y \cos(x^2) \, dy \, dx = \int_0^9 \frac{x}{2} \cos(x^2) \, dx = \frac{1}{4} \sin x^2 \bigg|_{x=9}^{x=0} = \frac{\sin 81}{4} \).
Problem 5. Set up a double integral in polar coordinates to find the volume of the solid that lies under the paraboloid \( z = x^2 + y^2 \), above the \( xy \)-plane, and inside the cylinder \( x^2 + y^2 = 2x \). Do not evaluate the integral.

Solution: In polar coordinates, \( x^2 + y^2 = 2x \) becomes \( r^2 = 2r \cos(\theta) \). Dividing out one \( r \) gives \( r = 2 \cos(\theta) \). In polar coordinates, \( z = x^2 + y^2 = r^2 \), so the integral is 

\[
\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{0}^{2 \cos(\theta)} r^2 \, r \, dr \, d\theta.
\]

Problem 6. Set up the triple integrals to find the \( x \)-coordinate of the center of mass of a solid of constant density that is bounded by the parabolic cylinder \( y^2 = x \) and the planes \( x = z \), \( z = 0 \), and \( x = 1 \). Do not evaluate the integrals.

Due to a typographical error on the exam, Problem 6 was not graded.

Solution: To find the mass, we integrate 

\[
m = \int_{-1}^{1} \int_{y^2}^{x} \int_{0}^{k} k \, dz \, dx \, dy.
\]

To find \( M_{yz} \), we integrate 

\[
\int_{-1}^{1} \int_{y^2}^{x} kx \, dz \, dx \, dy.
\]

\( \bar{x} = \frac{M_{yz}}{m} \).

Problem 7. Set up a triple integral in spherical coordinates to find \( \int \int \int_{E} x^2 \, dV \), where \( E \) lies between the spheres \( \rho = 1 \) and \( \rho = 3 \) and above the cone \( \phi = \pi/4 \). Do not evaluate the integral.

Solution: 

\[
\int_{0}^{2\pi} \int_{0}^{\pi/4} \int_{1}^{3} (\rho \sin \phi \cos \theta)^2 \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta.
\]

Problem 8. Evaluate \( \int_{C} (x - 2y^2) \, dy \), where \( C \) is the arc of the parabola \( y = x^2 \) from \((-2, 4)\) to \((1, 1)\).

Solution: \( x(t) = t, \ y(t) = t^2 \), so the integral is 

\[
\int_{-2}^{1} (t - 2(t^2)^2) \frac{1}{2} \, dt = 48.
\]
Problem 9. Evaluate the integral $\int_C \mathbf{F} \cdot \mathbf{dr}$, where $\mathbf{F}(x, y) = x^2 y^3 \mathbf{i} - y\sqrt{x} \mathbf{j}$ and $\mathbf{r}(t) = t^2 \mathbf{i} - t^3 \mathbf{j}$, $0 \leq t \leq 1$.

Solution: $\int_0^1 \mathbf{F}(\mathbf{r}(t)) \cdot d\mathbf{r} = \int_0^1 \langle (t^2)^2 (-t^3)^3, -(t^3)^2 \sqrt{t^2} \rangle \cdot \langle 2t, -3t^2 \rangle dt = \int_0^1 -2t^{14} - 3t^6 dt = \frac{-59}{105}$. 