Metric TVS topology comes from a metric.

Main Thm: Metrizable $\iff$ has a countable local base.

- $W$ balanced open balls
- @ origin
- $\&$ invariant

Additionally, if $V$ is locally convex, we can arrange that all open balls are convex.

The hard part, of course, is how to construct the metric.

Some comments first:

Recall we showed before that any nhbd of $0$ contains a balanced nhbd of $0$. Likewise, all convex nhbds contain a balanced convex nhbd of $0$.

If $U$ is an open nhbd of $0$, then $\exists U'$ balanced, $u \in U$, s.t., $U' + U' \subseteq U$, $U'$ open.

If locally convex, we may assume $U'$ is convex, at the expense
of making \( u' \) smaller.

Now assume \( \{ S_n \}_{n=1}^{\infty} \) is a local base. We showed before that there exists a balanced local base, \( \mathcal{U} \) in a locally convex TVS, a convex, balanced local base.

Let \( U_1 = S_1 \).

Let \( U_2 = S_2 \cap U_1 \),
\( U_3 = S_3 \cap U_2 \),
\( U_{n+1} = S_{n+1} \cap U_n \)
— where \( U' \) is the set given on the previous page.

Then each \( U_n \) is open, balanced (and convex if locally convex),
& since \( U_n \subseteq S_n \), \( S_3 \) is a local base, \( \mathcal{U} \cap U_3 \) is also a local base.

Furthermore \( U_{n+1} + U_{n+1} \subseteq U_n + U_n' \subseteq U_n \).

This proves claim (1) on p.18 of Rudin. I felt more needed to be said.

Now we describe the construction of the invariant metric. Recall that \( d(u,v) = d(0,u+v) \),
so first we need to describe the function \( f \).

\[ d(v_i, v_j) = f(v_i - v_j) \]

Consider rational numbers of the form

\[ \mathcal{N} = \sum_{n=1}^{\infty} c_n z^{-n} \]

where \( \mathcal{N} \) is arbitrary.

Such \( \mathcal{N} \) are between \( 0 \leq \mathcal{N} < 1 \).

Let \( A(r) = N \) if \( r \geq 1 \)

\[ A(r) = c_1 w_1 + c_2 w_2 + \cdots + c_n w_n \]

if \( \mathcal{N} \) has the above form.

Define \( f(x) = \inf \{ r : x \in A(r) \} \)

of that form.

I.e., if \( x \not\in \bigcup (w_1 + \cdots + w_n) \)

This means the metric \( d \leq 1 \),

but remember, we only really care about nearby points for defining the topology.

(We should note the minimum exists!)\(^1\)

(We should note it does for any subset of \( \mathbb{R}_{\geq 0} \).)

Note also for \( \mathbb{R}^n \), this describes the metric, at least for points less than distance \( f \) a point, \( \mathbb{E}^f \).
Proof of Thm 1. We need to show 3 properties to prove $d$ is a metric:

Property 1) \[ d(v,v) = f(0) = 0. \]

Property 2) \[ d(v,w) = f(v-w) \quad \text{in Hausdorff...} \]
\[ d(w,v) = f(w-v), \]
\[ \text{i.e., } f(v) = f(-v) \quad \text{obvious because each } V_n \text{ is balanced.} \]

Property 3) \( \Delta \)-inequality: \[ f(v+w) \leq f(v) + f(w) \]
[Of course, the hard one...]

We claim \[ A(r) + A(s) \leq A(r + s) \]
for \( r, s \) of the form \( t \).

This implies \( A(r) \leq A(r) + A(t-s) = A(t) \).

If \( f(x) + f(y) \geq 1 \), \[ \text{the } \Delta \text{-inequality is obvious. Otherwise, let } \epsilon > 0. \] Then \( \exists r, s \) of the form \( t \) such that \[ f(x) < r \quad x \in A(r), \]
\[ f(y) < s \quad y \in A(s) \implies x+y \in A(r+s), \]
\[ r+s < f(x)+f(y) + \epsilon. \]

So \[ f(x+y) \leq r+s < f(x)+f(y) + \epsilon. \] Take \( \epsilon = 0, \) get
\[ f(x+y) \leq f(x) + f(y). \]
Now we prove the claim \( A(r) + A(s) < A(r+s) \), (p. 5/11).

Remember, we still need to prove the open balls are balanced, a local base, (a convex...).

We prove \( A(r) + A(s) \leq A(r+s) \) by induction on \( N \geq 1 \)

When \( N=1 \), \( r, s = \text{Cor} \, \frac{1}{2} \)

\[
A(0) = \frac{3}{0} \quad (r,s) = (0,0) \quad \frac{3}{0} + \frac{3}{0} = \frac{3}{0}
\]

\[
A(\frac{1}{2}) = \frac{1}{0} \quad (r,s) = \frac{1}{0} + \frac{1}{0} = \frac{1}{0}
\]

Assume for \( N-1 \). Let \( r+s < 1 \),

\[
\begin{align*}
\sqrt{\sum_{n=1}^{N-1} c_n(r)z^{-n} + c_n(r)z^{-N}} &= r' + c_n(r)z^{-N} \\
\sum_{n=1}^{N-1} c_n(s)z^{-n} + c_n(s)z^{-N} &= s' + c_n(s)z^{-N}
\end{align*}
\]

\[
A(r) = A(r') + c_n(r)u^n
\]

\[
A(s) = A(s') + c_n(s)u^n
\]

\[
A(r) + A(s) \leq A(r') + A(s') + c_n(r)u^n + c_n(s)u^n
\]

\[
\leq A(r+s') + c_n(r)u^n + c_n(s)u^n
\]
If \( c_w(r) = c_w(s) = 0 \), \( r > r', s > s' \), done.

If \( c_w(r) = 0, c_w(s) = 1 \), \( r' = r \)

\[
A(r) + A(s) \leq A(r + s') + \ell_w
\]

\[
= A(r + s) \quad \text{(No carrying of the last bit)}
\]

Likewise, \( c_w(r) = 1, c_w(s) = 0 \), \( r > r', s > s' \), in the addition.

Finally, if \( c_w(r) = c_w(s) = 1 \)

\[
A(r) + A(s) \leq A(r + s') + (\ell_w + \ell_w)
\]

\[
= A(r + s) \quad \text{(Induction)}
\]

\[
\leq A(r' + s' + 2^{1-w}) = A(r + s).
\]

Finally, why a local base? Open balls

\[
B_\delta(c) = \bigcup_{r \leq \delta} A(r)
\]

So \( B_\delta(c) \subseteq \ell_w \) for \( \delta < 2^{-w} \).
Convexity is a little more subtle. 

\[ \{ U_n \} \text{ all convex} \Rightarrow A(r) \text{ are all convex} \]

\[ a \tau + A(r) + (1-\tau) A(r) \]

\[ = \sum C_n(r) \left( \tau U_n + (1-\tau) U_n \right) \]

\[ \leq \sum C_n(r) U_n = A(r), \]

Why is \( \bigcup A(r) \) convex?

We showed \( A(r) \leq A(t) \) for \( r, t \in [a, b] \), so this union is an increasing union of convex sets, hence convex (any 2 points lie in this after a finite amount of time).

New topic: Bounded Linear Operators.

Means: maps bounded sets into bounded sets.

(Not the same as "bounded function," since obviously we know most linear maps are not bounded.)
Thm. If \( \mathcal{L} : V \rightarrow W \) is a linear mapping of TVS's, then

\[
\forall c \in V \Rightarrow \exists L(b) \Rightarrow \text{the set } \{L(b_n)\}_{n=1}^{\infty} \text{ is bounded in } W \text{ for any sequence } v_n \rightarrow 0 \text{ in } V.
\] (c)

Moreover, if \( V \) is metrizable, then these 3 properties are equivalent, in particular also to the property

\[
\forall v \in V \rightarrow 0 \Rightarrow L(v_n) \rightarrow 0.
\] (d)

Proof. Assume (a), & let \( E \) be a bounded subset of \( V \).
We must show \( \mathcal{L}(E) \) is bounded.

Let \( U \) be a nbhd of 0 in \( W \).

\[
\forall c \in V \Rightarrow \exists \text{ open set } S \text{ containing } 0 \text{ in } V, \text{ s.t.,} \;

-2(b(S)) \subseteq U,
\]

\[
E \text{ bdd } \Rightarrow \text{ for large } t, \quad E \leq t \quad S
\]

Therefore \( \mathcal{L}(E) \leq L(t(S)) = L(S) \leq tU \text{ for large } t \quad (a) \Rightarrow (b)
\]

\( \mathcal{L}(E) \) is bounded, \( \Rightarrow \) (a) \( \Rightarrow \) (b) \( \checkmark \)
Assume (b), (c) follows if we can show convergent sequences are bad.

Indeed, let \( U \) be an open nbhd of \( 0 \).
\( U \) contains a balanced open nbhd \( U' \) of \( 0 \).
For large \( n \), \( v_n \in U' \) (def'n of convergence to 0).

As \( U \cap U' = U' \),
the finitely many points \( v_1, \ldots, v_n \) not in \( U' \) are contained in \( RU \) for some \( R > 1 \). \( U' \) is balanced, so \( U \) is balanced, \( RU \).

For all \( t \geq k \),
\( \exists v_n \in RU, tU \subseteq RU \), so by def of balanced,
\( \exists v_n \) is odd. So (b) \( \Rightarrow \) (c).

To prove the rest, now assume \( V \) is metrizable, we need to show (c) \( \Rightarrow \) (d) \( \& \) (c) \( \Rightarrow \) (a).

First we prove (d) \( \Rightarrow \) (a). Assume (a) is false, i.e., \( A \subseteq W \) s.t. \( A \cap U \) is not open.
$V$ is metrizable, so has a countable local base, none of whose members are subsets of $X^*(a)$.

Let $\{ S_1, S_2, \ldots \}$ be this local base.

Let $V_n \in S_n$ be an element not contained in $X^*(a)$.

Then $V_n \rightarrow c$ (by def'n) but $X(a)$ never gets inside $U$, so (d) is false.

Finally, we come to proving (c) $\Rightarrow$ (d)

$V_n \rightarrow c$ implies for each $k \geq 1$, $\exists N = N_k$ such that $d(V_n, c) \leq \frac{1}{k}$ for all $n \geq N_k$.

Let $c_n = k$ if $N_k \leq n < N_{k+1}$

Then $d(c_n, c) = \frac{1}{k}$ and $c_n \rightarrow c$.

So $\{c_n\}_{n \geq k}$ is bdd in $V$.

& by assumption of (c), $\{d(c_n, c)\} = \{c_n - c\}$ is bdd in $W$.

Let $U$ be any nhbd of $c$ in $W$ & $U'$ a balanced sub-nhd. $U' \subseteq U$ containing $c$.

Then for large $k$:

$\{c_n - c\} \leq t U$.
For \( n \) large, \( C_n \geq t \),

\[
c_{n \cdot c_l(v_n)} \in c \cdot U'
\]

\[
\lambda(v_n) \in \frac{t}{c} U' \subseteq U' \quad \text{since } U'
\]

is balanced.

U (arbitrary, so so \( \lambda(v_n) \in U \) for \( n \) large.

That means \( \lambda(v_n) \to \infty \).

\( \square \).