Lecture Notes II

Duals of Banach Spaces

Now we study $B(V, W) =$ odd linear maps $V \rightarrow W$ & its topologies.

Norm If $V, W$ normed spaces

$$||A|| = \sup_{||v|| = 1} ||Av|| = \sup_{||v|| \leq 1} ||Av||$$

exists since $||.||$ is cts for $x \in V \Rightarrow$ image is bdd.

$$||A|| = \sup_{V \neq 0} \frac{||Av||}{||v||}$$

Proposition $B(V, W)$ is a normed space under this norm, in fact a Banach space (complete normed space) if $W$ is a Banach space.

To show a normed space

If $x \neq 0$ we need to check $\forall x \neq 0$

1. $||x|| = ||x||$ (Obvious)

2. $||x + y|| \leq ||x|| + ||y||$

   (Proof: $||x|| = 1 \Rightarrow ||x + y|| \leq ||x|| + ||y||$)

4. $||x|| = 0 \iff x = 0$ (Obvious)
For the second part, we assume now that \( W \) is a Banach space (complete). If \( \{f_n\}_n \) is a Cauchy sequence of operators in the \( \|\cdot\| \) norm,

\[
\|f_n - f_m\| \to 0 \quad \text{for} \quad \|u\| \leq 1 \quad \text{as} \quad n, m \to \infty.
\]

\[\Rightarrow \exists f_n \text{ Cauchy, converges to limit}\]

\[\text{called} \quad f(u).\]

Now, by properties of limits & TVS,

\[f(v+w) = f(v) + f(w),\]

\[f(\alpha v) = \alpha f(v),\]

Hence \( f \) linear.

Let \( \epsilon > 0 \) & \( n, m \) large enough so that \( \|f_n - f_m\| \leq \epsilon \).

Now

\[
\|f - \text{limit}\| = \lim_{n \to \infty} \|f_n - f\| \\
\leq \lim_{n \to \infty} \|f_n - f_n\| + \|f_n - f_m\| + \|f_m - f\| \\
= \epsilon + \epsilon + \epsilon = 3\epsilon.
\]

\[\Rightarrow \|f - \text{limit}\| \leq 3\epsilon,\]

\[\Rightarrow f_n \to f \quad \text{in operator norm}.\]
Now we specialize to when \( W = k \).

**Bad (sometimes) notation:** Let \( v \in V \), \( \alpha \in V^* \) denote \( \langle v, \alpha \rangle \) by

\[
\langle v, \alpha \rangle.
\]

**Careful:** Not an inner product, just a pairing \( V \times V^* \rightarrow k \).

But it satisfies Cauchy-Schwarz: \( |\langle v, \alpha \rangle| \leq ||v|| \cdot ||\alpha|| \).

Given \( \alpha \in V^* \),

\[
||\alpha|| = \sup_{||v||=1} |\langle v, \alpha \rangle|,
\]

This norm makes \( V^* \) into a Banach space — strictly stronger topology than weak* if space is unital.

*Theorem:* Norm topology on \( V^* \) is stronger than weak* topology. They coincide if \( \dim V < \infty \), but norm topology is strictly stronger if \( \dim V = \infty \).

**Proof:** Let \( B^* = \) closed unit ball in \( V^* \) in norm topology.

Then we claim

\[
||\alpha|| = \sup_{\alpha \in B^*} |\langle v, \alpha \rangle|,
\]

i.e., usual \( \alpha \in B^* \) norm on \( V \) coincides with

\[
||v|| = \sup_{\alpha \in B^*} |\langle v, \alpha \rangle|.
\]
operator norm on \( V \) viewed as a subspace of \( V^* \), for all \( \lambda \in V \)

If \( \langle \lambda, \mu \rangle \leq \|\lambda\| \|\mu\| \)

but for all \( \mu \in B^* \)

\( \|\langle \lambda, \mu \rangle\| \leq \|\lambda\| \|\mu\| \)

so indeed \( \|\lambda\| = \sup \|\langle \lambda, \mu \rangle\| \)

where \( \mu \in B^* \)

Now, the map \( \lambda \mapsto \langle \lambda, \mu \rangle \), for any fixed \( \mu \in B^* \) is a bounded linear functional on \( V^* \) with norm \( \|\lambda\| \).

Thus weak* topology is weaker than norm topology.

If \( \lambda \in V = \mathbb{C}^\infty \) suppose weak* topology coincides with norm topology, then weak* topology is locally bounded. A set is weakly locally bounded if all limit points on it are bounded.

Let \( U \) be an open neighborhood in weak* topology.

\[ U = \bigcap_{\varepsilon > 0} \left\{ \lambda : \|\langle \lambda, \mu \rangle\| < \varepsilon \right\} \] for some \( \varepsilon > 0 \)
So any open null $U$ contains a subspace of infinite dimension. (Hence cannot have finite dim, since it has finite codim. & space is infinite.)

Let $\alpha \in \mathcal{W}$ be nontrivial.

Let $V$ a vector for which $\langle v, \alpha \rangle > 0$.

Thus $v$ is not odd on $U$.

$\Rightarrow v$ is not odd on $U$.

$\Rightarrow U$ is not weakly bounded, a contradiction.

Proposition In $V^*$, unit ball $B^*$ in norm topology is weak* opt.

Proof \[ \alpha \in B^* \iff |\langle v, \alpha \rangle| \leq 1 \quad \text{for all} \quad x \in \text{open unit ball } B \]

Banach-Alaoglu: \{ $x \in W$ \mid Mv \leq 1 \text{ for all } v \in V$ \} in $V$. \n
works for any open null $U$, \n
$B^*$ is weak* compact. \n
$\square$
Proposition

Let \( V \) be normed spaces and \( \mathcal{A} \subset B(V,W) \). Then

\[
\| A \| = \sup_{\| x \| \leq 1} \left| \langle A(x), y \rangle \right|
\]

\( \| x \| \leq 1 \).

Proof
We have, for any \( x \in V \)

\[
\| A(x) \| = \sup_{\| x \| \leq 1} \left| \langle A(x), y \rangle \right| \quad \text{by claim mpt of last Thm}
\]

so prop follows. \( \Box \).

Dual of Dual

Last Thm showed we have an isometric embedding \( V \subseteq (W^*)^* \), normalized TVS.

\( V \) Banach complete, isometric image is complete, hence closed, so we may view \( V \) as a closed subspace of \( (W^*)^* \).

Characterization of Image

Those sets of \( (W^*)^* \) which are UTVS in weak* topology,
Reflexive $V = V^{**}$

(more correctly, the image is onto)

$\iff$ weak$^*$ topologies coincide

and norm topology

induced by norm topology on $V$

self-induced from $V$

Annihilators

Let $V$ = Banach space

$M \subseteq V$ subspace

$N \subseteq V^*$ subspace

Annihilators: $M^\perp = \{ x \in V^* \mid \langle x, v \rangle = 0 \text{ for all } v \in M \}$

$N = \{ x \in V^* \mid \langle x, v \rangle = 0 \text{ for all } v \in N \}$

We put the $^\perp$ superscript on the left or right to distinguish between subspaces of $V^*$ and $V$, respectively.

$M^\perp$ is a weak$^*$-closed subspace, since it is the intersection of closed subspaces. M and $V^*$ are closed under the $\langle \cdot, \cdot \rangle$ map $v \mapsto \langle v, \cdot \rangle$.
\( M \) is a closed subspace, since each \( A_i \) is cts.

**Thm.** In this setup, \( \overline{\text{\( (M^+) \)}} = \text{closure of } M \text{ in } V \)

\[ \left( N^+ \right)^+ = \text{weak* closure of } N \text{ in } V^* \]

**Proof.**

First, \( v \in M \Rightarrow v \in \overline{\text{\( (M^+) \)}} \) obviously

\( \lambda \in \mathbb{C} \Rightarrow \lambda \in \left( N^+ \right)^+ \)

\[ \Rightarrow M \in \overline{\text{\( (M^+) \)}} \Rightarrow \overline{M} \in \overline{\text{\( (M^+) \)}} \]

\[ N \in \left( N^+ \right)^+ \Rightarrow \overline{N} \in \left( N^+ \right)^+ \]

we need to show the reverse inclusions.

If \( v \notin \overline{M} \), then since \( V \) is locally convex

\( M \) subspace, \( \exists \mathbb{B} \text{ cpt.} \)

in particular convex, balanced

\[ \exists \text{ convex subspace } M^+ \text{ s.t. } M \subseteq M^+ \subseteq V^* \]

Hahn-Banach theorem.
That means \( \nu \notin \overline{\nu(M)} \), so \( \overline{\nu} = \overline{\nu(M)} \).

Now assume \( \Lambda \notin \overline{\nu} = \text{weak} \nu \text{-closure of } N \) (not norm closure).

Again, \( \nu \) is locally convex so exists its iner-fil \( \nu \in V \) (i.e. \( \nu \in \mathcal{L} \)).

\( \text{Let } \Lambda \nu \in \mathcal{L}, \text{ but } \nu \in N, \text{ so } \overline{\nu} = (\overline{\nu})^+ \). \( \square \).

**Corollary**

Let \( M \) be a normed-closed subspace of \( V \), then \( M = \text{annihilator of its annihilator} \).

Likewise, \( N = \text{weak} \nu \text{-closed subspace of } V \), so \( N = \text{annihilator of its annihilator} \).