Dear Martin,

You asked about the one-variable Rankin-Selberg integral. Actually, it involves a limit. It uses the fact that horocycles are equidistributed on $SL_2(\mathbb{Z}) \backslash H$.

**Thm** Let $f$ be a smooth function on $H$ which is invariant under $SL_2(\mathbb{Z})$ and which decays rapidly ($O(y^{-N})$, any $N > 0$) in the fundamental domain $\mathcal{F} = \{ x+iy \mid \frac{i}{2} < x \leq \frac{1}{2}, |x+iy| \geq 1 \}$. Then

$$\lim_{y \to 0} \frac{1}{y^2} \int_{\mathcal{F}} f(x+iy) dx = \int_H f(x+iy) dxdy, \quad (\ast)$$

Weaker conditions on $f$ may be used. To prove this "horocyclic equidistribution" theorem, which is a first case of modern results of Margulis, Ratner, etc., on unipotent flows, we follow Weil's method. Since $f \in L^2(SL_2(\mathbb{Z}) \backslash H)$, $f$ has a spectral expansion

$$f = \langle f, \varphi_0 \rangle \varphi_0 + \sum_{j=0}^{\infty} \langle f, \varphi_j \rangle \varphi_j + \frac{1}{4\pi} \int_R \langle f, \varphi_0 \rangle \varphi_0 \int E_s(t) \, dt,$$

where $\varphi_0$ is the constant function $\frac{3}{\pi}$

$$\varphi_1, \varphi_2, \ldots$$

are Maass cusp forms.

$$E_s(t) = \frac{1}{2} \sum_{(m,n) \in \mathbb{Z}^2 \backslash \mathbb{G}} \left( \frac{N}{1+m^2+n^2} \right)^s$$

is the standard nonholomorphic Eisenstein series.
It suffices to show (4) for each of these basis functions. First, (4) is obvious for constants (the \( \pi/3 \) is to 
\[ \int_{\mathbb{R}} \frac{dx}{x^2/\pi^2} = \pi/3 \]). For the Maass cusp forms,

\[ \int_{\mathbb{R}} \varphi(x) dx = 0 \quad \text{for all} \ y \quad \text{by the definition of cuspidality,} \]

This is fortuitous, since these Maass forms are quite mysterious & transcendental, and allow us little to attack them with. Finally, the theory of the constant term tells us

\[ \int_0^\infty \psi(x) dx = \gamma + \frac{\pi^{1/2}}{\Gamma(1/2)} \zeta(1, s) \frac{1}{\Gamma(s/2)} \frac{1}{\pi^{1/2}} \frac{1}{\Gamma(s)} y^{-1/2} \]

\[ = O(\sqrt{y}) \quad \text{for } s \approx \frac{1}{2}. \]

Thus the theorem is proved. Note that the only effect of \( SL_2(\mathbb{Z}) \) itself appears through \( \psi(s) \) in the constant term of the Eisenstein series. In fact, Sarnak & Zagier have explained how the reformulate RH as a statement about the role of equidistribution.

Anyway, back to Rankin-Selberg. The \( L \)-function integral is given by

\[ \int_{\mathbb{R}} H(x + iy) \psi(x + iy) \frac{dx \, dy}{y^2}. \]
where

\[ H(z) = f_1(z) f_2(z) y^k \]

if \( f_1, f_2 \) are holomorphic cusp forms of weight \( k \)

or

\[ \Phi \gamma_1, \gamma_2 \] if \( \gamma_1, \gamma_2 \) are Maass cusp forms.

(One should also assume \( H \) is not odd, \(-H(z) \neq H(z)\) for some \( z \)) or else the integral vanishes.

Now \( H \) is smooth \& of rapid decay, hence \( H(x+iy) E_5(x+iy) \) is also, and since automorphic satisfies our theorem.

The Rankin-Selberg integral is thus

\[ \lim_{y \to 0} \frac{T}{3} \int_0^1 H(x+iy) E_5(x+iy) \, dx, \]

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