Solutions to the Practice Final Exam

(1) Find all complex solutions of the equation \( z^3 = 1 \).

The three solutions are the \( \cos \theta + i \sin \theta \) for \( \theta = 0, \theta = \frac{2\pi}{3}, \theta = 2 \cdot \frac{2\pi}{3} \). When we simplify \( \cos \theta + i \sin \theta \) for these three values of \( \theta \), we get the solutions \( z_1, z_2, z_3 \) where

\[
    z_1 = 1, \quad z_2 = -\frac{1}{2} + i \frac{\sqrt{3}}{2}, \quad z_3 = -\frac{1}{2} - i \frac{\sqrt{3}}{2}.
\]

(2) Solve the equation \( z^3 = -8i \), where \( z \) is a complex number.

We are asked to find the cube roots of \( -8i \). We can use the Theorem at the bottom of page 20 of the complex numbers notes. We have \( n = 3 \) in this case. The complex number \( -8i \) can be written

\[
    -8i = 8e^{i(3\pi/2)}.
\]

According to the Theorem on page 20, the cube roots are

\[
    8^{1/3} \left( \cos \left( \frac{3\pi/2 + 2k\pi}{3} \right) + i \sin \left( \frac{3\pi/2 + 2k\pi}{3} \right) \right) \text{ for } k = 0, 1, 2.
\]

This simplifies as

\[
    2i, \quad -\sqrt{3} - i, \quad \sqrt{3} - i
\]

and we have the answer to problem (2).

What we did here is just to plug into a formula. You may prefer to see an answer to problem (2) that is more conceptual in nature. The answer that you see below is longer than the plug-in answer, but it may help you to understand complex numbers. If we could find some solution \( z_0 \) to the equation \( z^3 = -8i \), then we would get all three solutions in the following way: Simply multiply the solution \( z_0 \) by the numbers \( z_1, z_2, z_3 \) that were found in problem (1). Here is the reason for this: We know from problem (1) that \( z_1, z_2, z_3 \) satisfy \( z_1^3 = 1, z_2^3 = 1, z_3^3 = 1 \). If we had a particular \( z_0 \) with the property \( z_0^3 = -8i \) then we would have

\[
    (z_0z_1)^3 = z_0^3z_1^3 = (-8i)(1) = -8i,
\]

\[
    (z_0z_2)^3 = z_0^3z_2^3 = (-8i)(1) = -8i,
\]

\[
    (z_0z_3)^3 = z_0^3z_3^3 = (-8i)(1) = -8i.
\]

Now we search for just one \( z_0 \) with the property \( z_0^3 = -8i \). As before,

\[
    -8i = 8e^{i(3\pi/2)}.
\]

If \( z_0 = 2e^{i(\pi/2)} \) then \( z_0^3 = 2^3(e^{i(\pi/2)})^3 = 8e^{i(3\pi/2)} = -8i \). Another way to write this \( z_0 \) is

\[
    z_0 = 2(\cos(\pi/2) + i \sin(\pi/2)) = 2i.
\]
Now we know that the three solutions to $z^3 = -8i$ are

$$z_0 = (2i)(1) = 2i, \quad z_1 = (2i)\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right), \quad z_2 = (2i)\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right).$$

These three solutions to $z^3 = -8i$ simplify as

$$2i, \quad -\sqrt{3} - i, \quad \sqrt{3} - i.$$

This gives the same answer to problem (2) that we had before.

(3) Evaluate $\int_3^4 \frac{dx}{x^2\sqrt{x^2 - 4}}$.

We use $x = 2\sec\theta$. Then

$$\sqrt{x^2 - 4} = \sqrt{4\sec^2\theta - 4} = \sqrt{4(\sec^2\theta - 1)} = \sqrt{4\tan^2\theta} = 2|\tan\theta| = 2\tan\theta,$$

where the last equality is justified by the fact that the required definite integral from $x = 3$ to $x = 4$ involves only $x$ with $x > 2$, and this means $0 < \theta < \pi/2$. Now we have

$$\int_3^4 \frac{dx}{x^2\sqrt{x^2 - 4}} = \frac{1}{4} \int \frac{2\sec\theta \tan\theta \, d\theta}{\sec^2\theta \cdot 2 \tan\theta} = \frac{1}{4} \int \frac{d\theta}{\sec\theta} = \frac{1}{4} \int \cos\theta \, d\theta = \frac{1}{4} \sin\theta + C$$

$$= \frac{1}{4} \tan\theta + C = \frac{1}{4} \cdot \left(\frac{1}{2}\sqrt{x^2 - 4}\right) + C = \frac{\sqrt{x^2 - 4}}{4x} + C.$$

Finally, we get

$$\int_3^4 \frac{dx}{x^2\sqrt{x^2 - 4}} = \left. \frac{\sqrt{x^2 - 4}}{4x} \right|_3^4.$$

(4) Find the Simpson approximation $S_4$ for $\int_1^3 \frac{dx}{x}$.

$$S_4 = \frac{0.5}{3} \left( \frac{1}{1.0} + \frac{4}{1.5} + \frac{2}{2.0} + \frac{4}{2.5} + \frac{1}{3.0} \right).$$

(5) Find the Maclaurin series of $\ln(1 + x^2)$. For what values of $x$ does this converge?

The Maclaurin series of $\frac{1}{1-x}$ is $\sum_{n=0}^{\infty} x^n$. If we replace $x$ by $-x$ then we find that the Maclaurin series of $\frac{1}{1+x}$ is $\sum_{n=0}^{\infty} (-x)^n = \sum_{n=0}^{\infty} (-1)^n x^n$. When we integrate both sides of

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n,$$

we find $\ln(1 + x) + C_1 = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1} + C_2$. Substituting $x = 0$
into this equation gives us $C_1 = C_2$. This means that $C_1$ and $C_2$ can be cancelled and we can write

$$\ln(1 + x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1}.$$ 

By the way, this is exactly the same as $\ln(1 + x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}$. If we replace $x$ by $x^2$ then we get

$$\ln(1 + x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n (x^2)^n+1}{n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{n+1}.$$ 

We see that $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{n+1}$ is the Maclaurin series of $\ln(1 + x^2)$. Now we will find the interval of convergence of this Maclaurin series. If $a_n = \frac{(-1)^n x^{2n+2}}{n+1}$ then

$$\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{(-1)^{n+1} x^{2(n+1)+2}}{(n+1) + 1} \right| \cdot \frac{n+1}{(-1)^n x^{2n+2}} = \lim_{n \to \infty} \frac{n+1}{n+2} x^2 = x^2.$$

According to the ratio test, the Maclaurin series converges for $-1 < x < 1$ and diverges for both $x < -1$ and $x > 1$. We have to check the endpoints $x = -1$ and $x = 1$. In both cases, the alternating series test tells us that the series converges. This means that the Maclaurin series converges for $-1 \leq x \leq 1$.

(6) Find the interval of convergence of the power series $\sum_{n=1}^{\infty} \sqrt{n}(x + 2)^n$.

If $a_n = \sqrt{n}(x + 2)^n$ then

$$\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{\sqrt{n + 1}(x + 2)^{n+1}}{\sqrt{n}(x + 2)^n} \right| = \lim_{n \to \infty} \sqrt{\frac{n+1}{n}} |x + 2| = |x + 2|.$$ 

The ratio test tells us that the power series converges if $|x + 2| < 1$ and diverges if $|x + 2| > 1$. In other words, we have convergence on $(-3, -1)$, divergence on $(-\infty, -3)$ and divergence on $(1, \infty)$. We have to check the endpoints $-3$ and $-1$. When $x = -1$ the series is $\sum_{n=1}^{\infty} \sqrt{n}$, which diverges because $\sqrt{n}$ does not approach 0. When $x = -3$ the series is $\sum_{n=1}^{\infty} (-1)^n \sqrt{n}$, which diverges because $(-1)^n \sqrt{n}$ does not approach 0. All this means that the interval of convergence is precisely $(-3, -1)$.

(7) Find the area of the sector bounded by the polar curve $r = \sin \theta$ and the two rays $\theta = \pi/3$ and $\theta = \pi/2$. 

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The area is

\[
\frac{1}{2} \int_{\pi/3}^{\pi/2} \sin^2 \theta \, d\theta = \frac{1}{2} \int_{\pi/3}^{\pi/2} \frac{1}{2} (1 - \cos(2\theta)) \, d\theta = \frac{1}{4} (\theta - (1/2) \sin(2\theta)) \bigg|_{\pi/3}^{\pi/2}
\]

\[
= \frac{1}{4} (\pi/2) - \frac{1}{4} (\pi/3 - (1/2)(\sqrt{3}/2))
\]

(8) Find the arc length of the graph of \( f(x) = \ln(\cos x) \) over the interval \([0, \pi/4]\).

We know

\[1 + (f'(x))^2 = 1 + \left(\frac{-\sin x}{\cos x}\right)^2 = 1 + (-\tan x)^2 = 1 + \tan^2 x = \sec^2 x.\]

Therefore, the required arc length is

\[
\int_0^{\pi/4} \sqrt{1 + (f'(x))^2} \, dx = \int_0^{\pi/4} \sqrt{\sec^2 x} \, dx = \int_0^{\pi/4} \sec x \, dx = \ln |\tan x + \sec x| \bigg|_0^{\pi/4}
\]

\[= \ln(1 + \sqrt{2}) - \ln(1) = \ln(1 + \sqrt{2}).\]

(9) Prove that \( \sum_{n=1}^{\infty} \frac{\cos(n^3)}{n^{3/2}} \) converges.

First, we prove that the given series converges absolutely. The inequalities

\[0 \leq \left| \frac{\cos(n^3)}{n^{3/2}} \right| \leq \frac{1}{n^{3/2}}\]

and the convergence of \( \sum_{n=1}^{\infty} \frac{1}{n^{3/2}} \) tell us that the direct comparison test can be used to prove that \( \sum_{n=1}^{\infty} \left| \frac{\cos(n^3)}{n^{3/2}} \right| \) converges. This says that \( \sum_{n=1}^{\infty} \frac{\cos(n^3)}{n^{3/2}} \) converges absolutely. Now we conclude that \( \sum_{n=1}^{\infty} \frac{\cos(n^3)}{n^{3/2}} \) converges, since absolute convergence implies convergence.

(10) Evaluate \( \int \tan^{-1} x \, dx. \)

Since \( \frac{d}{dx} \tan^{-1} x = \frac{1}{1 + x^2} \), integration by parts gives

\[
\int \tan^{-1} x \, dx = x \tan^{-1} x - \int x \frac{1}{1 + x^2} \, dx.
\]
Now we use \( u = 1 + x^2 \), \((1/2)du = x \, dx\) to find

\[
x \tan^{-1} x - \int x \frac{1}{1 + x^2} \, dx = x \tan^{-1} x - \int \frac{(1/2) \, du}{u} = x \tan^{-1} x - \frac{1}{2} \ln u + C
\]

\[
= x \tan^{-1} x - \frac{1}{2} \ln(1 + x^2) + C.
\]

Combining all this, we obtain

\[
\int \tan^{-1} x \, dx = x \tan^{-1} x - \frac{1}{2} \ln(1 + x^2) + C.
\]

(11) Prove that \( \sum_{n=1}^{\infty} \sqrt[3]{\frac{n}{n^5 + n^4 + 3}} \) diverges.

We do a limit comparison with the series \( \sum_{n=1}^{\infty} \sqrt[2]{\frac{1}{n^2}} \). The limit of ratios is

\[
\lim_{n \to \infty} \sqrt[3]{\frac{n^3}{n^5 + n^4 + 3}} / \sqrt[2]{\frac{1}{n^2}} = \lim_{n \to \infty} \sqrt[5]{\frac{n^5}{n^5 + n^4 + 3}} = \sqrt[5]{1} = 1.
\]

Since 1 is positive and finite, the answer to the convergence/divergence question about the original series is the same as the answer to the convergence/divergence question about the new series \( \sum_{n=1}^{\infty} \sqrt[2]{\frac{1}{n^2}} \). Since the new series is actually \( \sum_{n=1}^{\infty} \frac{1}{n} \), which is the divergent harmonic series, we know that the original series diverges, also.

(12) Identify the curve with polar equation \( r = 4 \sin \theta \).

Multiplying both sides by \( r \), we get \( r^2 = 4r \sin \theta \). This is just \( x^2 + y^2 = 4y \), which can be rewritten \( (x - 0)^2 + (y - 2)^2 = 2^2 \). This is an equation of a circle in the \( xy \)-plane.

(13) Find the volume of the solid obtained by rotating the region bounded by \( y = 1 + x \), \( y = 1 - x \), \( x = 2 \) about the \( y \)-axis.

We use the Method of Shells. The volume is

\[
2\pi \int_0^2 x((1 + x) - (1 - x)) \, dx = 2\pi \int_0^2 2x^2 \, dx = 2\pi \left[ \frac{2x^3}{3} \right]_0^2 = \frac{32\pi}{3}.
\]

(14) Evaluate \( \int \tan x \sec^3 x \, dx \).

If \( u = \sec x \) then \( du = \tan x \sec x \, dx \) and

\[
\int \tan x \sec^3 x \, dx = \int (\sec^2 x)(\tan x \sec x) \, dx = \int u^2 \, du = \frac{u^3}{3} + C = \frac{\sec^3 x}{3} + C.
\]
(15) Evaluate \( \int \frac{6x^2 + 3x + 8}{x(x^2 + 4)} \, dx \).

We seek \( A, B, \) such that \( \frac{6x^2 + 3x + 8}{x(x^2 + 4)} = \frac{A}{x} + \frac{Bx + C}{x^2 + 4} \). We need

\[
\frac{6x^2 + 3x + 8}{x(x^2 + 4)} = \frac{A(x^2 + 4) + x(Bx + C)}{x(x^2 + 4)}.
\]

This is \( 6x^2 + 3x + 8 = A(x^2 + 4) + x(Bx + C) \). Plugging \( x = 0 \) into the above identity, we get \( 8 = A(4) \), which is \( A = 2 \). Now the identity \( 6x^2 + 3x + 8 = A(x^2 + 4) + x(Bx + C) \) simplifies to \( 6x^2 + 3x + 8 = 2(x^2 + 4) + x(Bx + C) \), which is \( 4x^2 + 3x = x(Bx + C) \), which is \( 4x^2 + 3x = Bx^2 + Cx \). Now we know \( B = 4 \) and \( C = 3 \), in addition to \( A = 2 \).

The original problem now becomes evaluating \( \int \frac{2}{x} + \frac{4x + 3}{x^2 + 4} \, dx \). The easy parts are \( \int \frac{2}{x} \, dx = 2 \ln |x| + C \) and \( \int \frac{4x}{x^2 + 4} \, dx = 2 \ln(x^2 + 4) + C \). For the more difficult part \( \int \frac{3}{x^2 + 4} \, dx \) we use \( x = 2 \tan \theta \), which gives us

\[
\int \frac{3}{x^2 + 4} \, dx = 3 \int \frac{2 \sec^2 \theta \, d\theta}{4 \sec^2 \theta} = \frac{3}{2} \int 1 \, d\theta = \frac{3}{2} \theta + C = \frac{3}{2} \tan^{-1} \left( \frac{x}{2} \right) + C.
\]

(16) Find the surface area of the surface obtained by rotating the curve \( c(t) = (\sin^3 t, \cos^3 t) \) about the \( x \)-axis for \( \pi/6 \leq t \leq \pi/3 \).

Let \( x(t) = \sin^3 t, \ y(t) = \cos^3 t \). We get \( x'(t) = 3 \sin^2 t \cos t, \ y'(t) = -3 \cos^2 t \sin t \). Then

\[
(x'(t))^2 + (y'(t))^2 = 9 \sin^4 t \cos^2 t + 9 \cos^4 t \sin^2 t = 9(\sin^2 t + \cos^2 t)(\sin^2 t \cos^2 t) = 9 \sin^2 t \cos^2 t.
\]

The required surface area is

\[
2\pi \int_{\pi/6}^{\pi/3} \cos^3 t \sqrt{(x'(t))^2 + (y'(t))^2} \, dt = 2\pi \int_{\pi/6}^{\pi/3} \cos^3 t (3 \sin t \cos t) \, dt
\]

\[
= 6\pi \left. \cos^4 t \sin t \right|_{\pi/6}^{\pi/3} = -6\pi \cos^5 t \bigg|_{\pi/6}^{\pi/3}
\]

\[
= \frac{6\pi}{5} \left( (\sqrt{3}/2)^5 - (1/2)^5 \right).
\]