Solutions to Practice Midterm 2

(1) Do either (a) or (b): (a) Prove that \( \sum_{n=0}^{\infty} \frac{x^{3n}}{n!} \) converges for all real \( x \). Find a simple formula for the sum \( \sum_{n=0}^{\infty} \frac{x^{3n}}{n!} \). (b) Prove that \( \sum_{n=0}^{\infty} \frac{(-1)^n x^{6n}}{(2n)!} \) converges for all real \( x \). Find a simple formula for the sum \( \sum_{n=0}^{\infty} \frac{(-1)^n x^{6n}}{(2n)!} \).

Solution to (1)(a): Use the ratio test:

\[
\lim_{n \to \infty} \left| \frac{x^{3(n+1)}}{(n+1)!} \right| / \left| \frac{x^{3n}}{n!} \right| = \lim_{n \to \infty} \frac{n!}{(n+1)!} \cdot \frac{x^{3n}}{x^{3(n+1)}} = \lim_{n \to \infty} \frac{|x|^3}{n+1} = 0 < 1.
\]

This shows that \( \sum_{n=0}^{\infty} \frac{x^{3n}}{n!} \) converges. If we replace \( x \) by \( x^3 \) in \( \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x \) then we get \( \sum_{n=0}^{\infty} \frac{x^{3n}}{n!} = e^{x^3} \).

Solution to (1)(b): Use the ratio test and \( \frac{(2(n+1))! = (2n+2)! = (2n+1)(2n+2)!}{(2n)!} \):

\[
\lim_{n \to \infty} \left| \frac{(-1)^{n+1} x^{6(n+1)}}{(2(n+1))!} \right| / \left| \frac{(-1)^n x^{6n}}{(2n)!} \right| = \lim_{n \to \infty} \frac{(2n)!}{(2n+1)(2n+2)} \cdot \frac{|x|^6}{(-1)^n x^{6n}} = \lim_{n \to \infty} \frac{|x|^6}{(2n+1)(2n+2)} = 0 < 1.
\]

This shows that \( \sum_{n=0}^{\infty} \frac{(-1)^n x^{6n}}{(2n)!} \) converges. If we replace \( x \) by \( x^3 \) in \( \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = \cos x \) then we get \( \sum_{n=0}^{\infty} \frac{(-1)^n x^{6n}}{(2n)!} = \cos(x^3) \).

(2) Let \( f(x) \) be a continuous function defined on the interval \([1, 9]\) such that \( f(1) = 4, f(2) = -2, f(3) = 2, f(4) = 3, f(5) = -2, f(6) = 1, f(7) = -3, f(8) = 3, f(9) = -4 \). Find the approximations \( T_4, M_4 \) and \( S_4 \) to \( \int_1^9 f(x) \, dx \). Recall that \( T \) stands for Trapezoidal, \( M \) stands for Midpoint and \( S \) stands for Simpson. Now assume that \( f(x) \) has a continuous second derivative and the additional property \(-30 < f''(x) < 20 \) for all \( x \) in the interval \([1, 9]\). What estimates can we make for the errors associated with \( T_4 \) and \( M_4 \)?

Solution to (2): \( \Delta x = 2 \) in this problem.

\[
T_4 = \frac{2}{2} (f(1) + 2f(3) + 2f(5) + 2f(7) + f(9)) = -6
\]
\[
M_4 = 2 (f(2) + f(4) + f(6) + f(8)) = 10
\]
\[
S_4 = \frac{2}{3} (f(1) + 4f(3) + 2f(5) + 4f(7) + f(9)) = -\frac{16}{3}
\]
We can choose $K_2 = 30$ because $-30 < f''(x) < 20$ implies $|f''(x)| < 30$. The error associated with $T_4$ is at most $\frac{30(9 - 1)^3}{12 \cdot 4^2} = 80$. The error associated with $M_4$ is at most $\frac{30(9 - 1)^3}{24 \cdot 4^2} = 40$.

(3) Determine whether $\sum_{n=2}^{\infty} \frac{2^n}{3^n - 5}$ converges or diverges.

Solution to (3): We use the limit comparison test with $\sum_{n=2}^{\infty} \frac{2^n}{3^n}$. We have

$$\lim_{n \to \infty} \frac{\frac{2^n}{3^n - 5}}{\frac{2^n}{3^n}} = \lim_{n \to \infty} \frac{3^n - 5}{3^n} = \lim_{n \to \infty} 1 - \frac{5}{3^n} = 1.$$ 

Since this limit is positive and finite, the answer to the convergence/divergence question about $\sum_{n=2}^{\infty} \frac{2^n}{3^n - 5}$ is the same as the answer to the convergence/divergence question about $\sum_{n=2}^{\infty} \frac{2^n}{3^n}$. We know that $\sum_{n=2}^{\infty} \frac{2^n}{3^n}$ converges, since $\sum_{n=2}^{\infty} \frac{2^n}{3^n} = \sum_{n=2}^{\infty} \left(\frac{2}{3}\right)^n$, which is a convergent geometric series (because $\left|\frac{2}{3}\right| < 1$). This implies that $\sum_{n=2}^{\infty} \frac{2^n}{3^n - 5}$ converges.

(4) A surface is formed by rotating the curve $y = (x+2)^3$, $0 \leq x \leq 1$ about the $x$-axis. Find the area of this surface. Hint: Evaluate the integral using an appropriate substitution.

Solution to (4): The surface area is

$$\int_{0}^{1} 2\pi(x+2)^3 \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx = \int_{0}^{1} 2\pi(x+2)^3 \sqrt{1 + (3(x+2)^2)^2} \, dx$$

$$= \int_{0}^{1} 2\pi(x+2)^3 \sqrt{1 + 9(x+2)^4} \, dx.$$ 

If we use the substitution $u = 1 + 9(x+2)^4$ then we get $(1/36)du = (x+2)^3 \, dx$. This gives

$$\int 2\pi(x+2)^3 \sqrt{1 + 9(x+2)^4} \, dx = \int 2\pi(1/36)u^{1/2} \, du = 2\pi(1/36)(2/3)u^{3/2} + C$$

$$= 2\pi(1/36)(2/3)(1 + 9(x+2)^4)^{3/2} + C.$$ 

The area is $2\pi(1/36)(2/3)(1 + 9(x+2)^4)^{3/2} \bigg|_{0}^{1} = (\pi/27)(730^{3/2} - 145^{3/2})$.

(5) Determine whether $\sum_{n=1}^{\infty} \tan(1/n)$ converges or diverges. Hint: One way to to do this is to start with the evaluation of $\lim_{n \to \infty} \frac{\tan(1/n)}{1/n}$. Is there another way?
Solution to (5): L'Hôpital's Rule gives
\[
\lim_{x \to 0} \frac{\tan x}{x} = \lim_{x \to 0} \frac{\sec^2 x}{1} = \sec^2 0 = 1.
\]
From this and \(\lim_{n \to \infty} (1/n) = 0\) we can conclude
\[
\lim_{n \to \infty} \frac{\tan(1/n)}{1/n} = \lim_{x \to 0} \frac{\tan x}{x} = 1.
\]
Since this limit is positive and finite, the limit comparison test tells us that the answer to the convergence/divergence question about \(\sum_{n=1}^{\infty} \tan(1/n)\) is the same as the answer to the convergence/divergence question about \(\sum_{n=1}^{\infty} 1/n\). We know that the harmonic series \(\sum_{n=1}^{\infty} 1/n\) diverges. This implies that \(\sum_{n=1}^{\infty} \tan(1/n)\) diverges. Another way to do problem (5) is to use the direct comparison test: The fact \(x < \tan x\) for \(0 < x < \pi/2\) implies \(1/n < \tan(1/n)\) for \(n = 1, 2, 3, \ldots\). Since the harmonic series \(\sum_{n=1}^{\infty} 1/n\) diverges, and we have \(0 < 1/n < \tan(1/n)\), we conclude that \(\sum_{n=1}^{\infty} \tan(1/n)\) diverges.

(6) Find \(\sum_{n=1}^{\infty} \frac{1}{3^{2n+1}}\). Your answer must be a simple number.

Solution to (6): The formula \(1 + c + c^2 + c^3 + \cdots = \frac{1}{1-c}\) is valid when \(|c| < 1\). We use this formula with \(c = \frac{1}{9}\) at the end of the following computation:
\[
\sum_{n=1}^{\infty} \frac{1}{3^{2n+1}} = \sum_{n=1}^{\infty} \frac{1}{3^{2n}3} = \frac{1}{3} \sum_{n=1}^{\infty} \frac{1}{3^{2n}} = \frac{1}{3} \sum_{n=1}^{\infty} \frac{1}{(3^2)^n} = \frac{1}{3} \sum_{n=1}^{\infty} \frac{1}{9^n} = \frac{1}{3} \sum_{n=1}^{\infty} \left(\frac{1}{9}\right)^n = \frac{1}{3} \cdot \frac{1}{9} \sum_{n=1}^{\infty} \left(\frac{1}{9}\right)^{n-1} = \frac{1}{27} \left(1 + \frac{1}{9} + \left(\frac{1}{9}\right)^2 + \left(\frac{1}{9}\right)^3 + \cdots\right) = \frac{1}{27} \cdot \frac{1}{1 - \frac{1}{9}} = \frac{1}{24}.
\]

(7) Find the interval of convergence of \(\sum_{n=1}^{\infty} \frac{(x - 2)^n}{\sqrt{n}}\). In this particular example, there is one and only one value of \(x\) such that the given power series converges conditionally at \(x\). Find that value of \(x\).
Solution to (7): When we use the ratio test with \(a_n = \frac{(x-2)^n}{\sqrt{n}}\), we get
\[
\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{(x-2)^{n+1}}{\sqrt{n+1}} \right| \cdot \frac{\sqrt{n}}{(x-2)^n} = |x-2| \frac{\sqrt{n}}{\sqrt{n+1}}.
\]

Now
\[
\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} |x-2| \frac{\sqrt{n}}{\sqrt{n+1}} = |x-2| \lim_{n \to \infty} \frac{n}{\sqrt{n+1}} = |x-2| \sqrt{1} = |x-2|.
\]

If \(|x-2| < 1\) then the power series converges absolutely. If \(|x-2| > 1\) then the power series diverges. This can be rewritten in the following way: If \(1 < x < 3\) then the power series converges absolutely. If \(x < 1\) or \(3 < x\) then the power series diverges. This takes care of every \(x\) except \(x = 1\) and \(x = 3\). If \(x = 1\) then the power series is
\[
\sum_{n=1}^{\infty} \frac{(1-2)^n}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}},
\]
which converges by the alternating series test (because \(\frac{1}{\sqrt{n}}\) is positive, decreasing with limit 0). If \(x = 3\) then the power series is
\[
\sum_{n=1}^{\infty} \frac{(3-2)^n}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}},
\]
which diverges because it is a \(p\) series with \(p = 1/2\). Now we know that the interval of convergence is \([1, 3)\). Here is the reason why this power series converges conditionally at \(x = 1\): We are looking at the series \(\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}\). As we have seen, the series converges.

However, the series \(\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}\) diverges because it is the series \(\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}\), which diverges, as we have seen already.

(8) Determine whether \(\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^{3/2}}\) converges or diverges.

Solution to (8): We use the integral test. The substitution \(u = \ln x\), \(du = (1/x)dx\) gives
\[
\int \frac{dx}{x(\ln x)^{3/2}} = \int \frac{du}{u^{3/2}} = \int u^{-3/2} du = -2u^{-1/2} + C = -2 \sqrt{\ln x} + C.
\]

Now
\[
\int_{2}^{\infty} \frac{dx}{x(\ln x)^{3/2}} = \lim_{b \to \infty} \int_{2}^{b} \frac{dx}{x(\ln x)^{3/2}} = \lim_{b \to \infty} \left( -2 \sqrt{\ln b} + 2 \sqrt{\ln 2} \right) = \frac{2}{\sqrt{\ln 2}},
\]
which shows that \(\int_{2}^{\infty} \frac{dx}{x(\ln x)^{3/2}}\) converges. Therefore, \(\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^{3/2}}\) converges.
(9) Determine whether \( \sum_{n=1}^{\infty} \left( \frac{n}{2n+1} \right)^n \) converges or diverges. Try to do this problem in two different ways.

Solution to (9): We use the root test. We compute

\[
\lim_{n \to \infty} \left| \left( \frac{n}{2n+1} \right)^n \right|^{1/n} = \lim_{n \to \infty} \left( \frac{n}{2n+1} \right)^n = \lim_{n \to \infty} \frac{n}{2n+1} = \frac{1}{2}.
\]

Since \( \frac{1}{2} < 1 \), we conclude that \( \sum_{n=1}^{\infty} \left( \frac{n}{2n+1} \right)^n \) converges. Another way to do problem (9) is to use the direct comparison test: We have

\[
0 < \left( \frac{n}{2n+1} \right)^n < \left( \frac{n}{2n} \right)^n = \left( \frac{1}{2} \right)^n
\]

and we know that \( \sum_{n=1}^{\infty} \left( \frac{1}{2} \right)^n \) is a geometric series that converges (because \(|1/2| < 1\)).

(10) Determine whether \( \sum_{n=1}^{\infty} \frac{(2n)!/(2n)!}{n!(3n)!} \) converges or diverges.

Solution to (10): We use the ratio test. We find

\[
\frac{(2(n+1))!(2(n+1))!}{(n+1)!(3(n+1))!} \cdot \frac{(2n)!(2n)!}{n!(3n)!} = \frac{(2n+1)!/(2n+1)!}{n!(3n)!} \cdot \frac{n!(3n)!}{(n+1)!(3n+1)!} \cdot \frac{(2n+1)!/(2n+1)!}{(2n)!/(2n)!} = \frac{2n+1}{n+1} \cdot \frac{2n+2}{3n+1} \cdot \frac{2n+1}{3n+2} \cdot \frac{2n+2}{3n+3}.
\]

This implies

\[
\lim_{n \to \infty} \frac{(2(n+1))!(2(n+1))!}{(n+1)!(3(n+1))!} \cdot \frac{(2n)!(2n)!}{n!(3n)!} = \lim_{n \to \infty} \frac{2n+1}{n+1} \cdot \frac{2n+2}{3n+1} \cdot \frac{2n+1}{3n+2} \cdot \frac{2n+2}{3n+3} = \frac{2 \cdot 2 \cdot 2 \cdot 2}{1 \cdot 3 \cdot 3 \cdot 3} = \frac{16}{27} < 1.
\]

Therefore, \( \sum_{n=1}^{\infty} \frac{(2n)!/(2n)!}{n!(3n)!} \) converges.