Solutions to Practice Midterm 1

(1) When we consider \( \int \sin^5 x \cos^4 x \, dx \), we see

\[
\int \sin^5 x \cos^4 x \, dx = \int \sin^4 x \cos^4 x \sin x \, dx = \int (\sin^2 x)^2 \cos^4 x \sin x \, dx = \int (1 - \cos^2 x)^2 \cos^4 x \sin x \, dx.
\]

This suggests the substitution \( u = \cos x \), \( du = -\sin x \, dx \), \( -du = \sin x \, dx \), which leads to

\[
\int \sin^5 x \cos^4 x \, dx = \int (1 - \cos^2 x)^2 \cos^4 x \sin x \, dx = -\int (1 - u^2)^2 u^4 \, du = -\int (1 - 2u^2 + u^4)u^4 \, du = -\int u^5 - \frac{2u^7}{7} - \frac{u^9}{9} \, du + C = -\frac{\cos^5 x}{5} + \frac{2\cos^7 x}{7} - \frac{\cos^9 x}{9} + C.
\]

The \( \sqrt{9 + x^2} \) in \( \int \frac{x^3 \, dx}{\sqrt{9 + x^2}} \) suggests the trigonometric substitution \( x = 3 \tan \theta \), \( dx = 3 \sec^2 \theta \, d\theta \), \( \sqrt{9 + x^2} = 3 \sec \theta \), which leads to

\[
\int \frac{x^3 \, dx}{\sqrt{9 + x^2}} = \int \frac{3^3 \tan^3 \theta \cdot 3 \sec^2 \theta \, d\theta}{3 \sec \theta} = 27 \int \tan^3 \theta \sec \theta \, d\theta = 27 \int \tan^2 \theta \tan \theta \sec \theta \, d\theta = 27 \int (\sec^2 \theta - 1) \tan \theta \sec \theta \, d\theta = 27 \int \sec^2 \theta \tan \theta \sec \theta \, d\theta - 27 \int \tan \theta \sec \theta \, d\theta = 9 \sec^3 \theta - 27 \sec \theta + C = \frac{1}{3} \left( \sqrt{9 + x^2} \right)^3 - 9\sqrt{9 + x^2} + C = \frac{1}{3} (9 + x^2)^{3/2} - 9(9 + x^2)^{1/2} + C.
\]

In case you were wondering how we got \( 27 \int \sec^2 \theta \tan \theta \sec \theta \, d\theta = 9 \sec^3 \theta + C \), we used the substitution \( u = \sec \theta \), \( du = \sec \theta \tan \theta \, d\theta \) to compute

\[
27 \int \sec^2 \theta \tan \theta \sec \theta \, d\theta = 27 \int u^2 \, du = 9u^3 + C = 9 \sec^3 \theta + C.
\]
(2) Two integrations by parts produce

\[ \int e^{2x} \cos(3x) \, dx = \left( \frac{1}{2} e^{2x} \right) \cos(3x) - \int \frac{1}{2} e^{2x} (-3 \sin(3x)) \, dx \]

\[ = \frac{1}{2} e^{2x} \cos(3x) + \frac{3}{2} \int e^{2x} \sin(3x) \, dx \]

\[ = \frac{1}{2} e^{2x} \cos(3x) + \frac{3}{4} e^{2x} (3 \sin(3x)) - \frac{9}{4} \int e^{2x} \cos(3x) \, dx . \]

After moving the \( \frac{9}{4} \int e^{2x} \cos(3x) \, dx \) term to the left side of the equation, we obtain

\[ \int e^{2x} \cos(3x) \, dx + \frac{9}{4} \int e^{2x} \cos(3x) \, dx = \frac{1}{2} e^{2x} \cos(3x) + \frac{3}{4} e^{2x} \sin(3x) + C , \]

which is

\[ \frac{13}{4} \int e^{2x} \cos(3x) \, dx = \frac{1}{2} e^{2x} \cos(3x) + \frac{3}{4} e^{2x} \sin(3x) + C . \]

Multiplying both sides by 4/13, we finally obtain

\[ \int e^{2x} \cos(3x) \, dx = \frac{2}{13} e^{2x} \cos(3x) + \frac{3}{13} e^{2x} \sin(3x) + C . \]

(3) Two integrations by parts give

\[ \int (\ln x)^{2} \, dx = x(\ln x)^{2} - \int x \left( 2(\ln x) \frac{1}{x} \right) \, dx = x(\ln x)^{2} - 2 \int \ln x \, dx \]

\[ = x(\ln x)^{2} - 2 \left( x \ln x - \int \frac{1}{x} \, dx \right) = x(\ln x)^{2} - 2x \ln x + 2 \int 1 \, dx \]

\[ = x(\ln x)^{2} - 2x \ln x + 2x + C . \]

Since \( x^3 - x^2 = x^2(x - 1) \), partial fractions applied to \( \int \frac{4x^2 + 5x + 2}{x^3 - x^2} \, dx \) require us to solve for \( A, B, C \) in the equation

\[ \frac{4x^2 + 5x + 2}{x^2(x - 1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x - 1} . \]

This is the same as solving for \( A, B, C \) in the equation

\[ \frac{4x^2 + 5x + 2}{x^2(x - 1)} = \frac{Ax(x - 1) + B(x - 1) + Cx^2}{x^2(x - 1)} , \]
which reduces to solving for $A$, $B$, $C$ in the equation

$$4x^2 + 5x + 2 = Ax(x - 1) + B(x - 1) + Cx^2.$$ 

When we substitute $x = 0$ in the last equation, we get $B = -2$. If we substitute $x = 1$ instead, then we get $C = 11$. Equating the $x^2$ terms in the equation $4x^2 + 5x + 2 = Ax(x - 1) + B(x - 1) + Cx^2$ leads to $4 = A + C$. Since $C = 11$, we get $A = -7$. Now that we have explicit values for $A$, $B$ and $C$, we can write

$$\int \frac{4x^2 + 5x + 2}{x^2 - x^2} \, dx = \int \frac{4x^2 + 5x + 2}{x^2(x - 1)} \, dx$$

$$= \int \frac{-7}{x} + \frac{-2}{x^2} + \frac{11}{x - 1} \, dx = -7 \ln |x| + \frac{2}{x} + 11 \ln |x - 1| + C.$$ 

Completing the square and using the substitution $u = (x - 1)/3$, $du = (1/3) \, dx$, $dx = 3 \, du$ we obtain

$$\int \frac{dx}{10 - 2x + x^2} = \int \frac{dx}{9 + (x - 1)^2} = \frac{1}{9} \int \frac{dx}{1 + ((x - 1)/3)^2}$$

$$= \frac{1}{3} \int \frac{du}{1 + u^2} = \frac{1}{3} \tan^{-1} u + C = \frac{1}{3} \tan^{-1} \left( \frac{x - 1}{3} \right) + C.$$ 

Now $\int_{1}^{\infty} \frac{dx}{10 - 2x + x^2} = \lim_{t \to \infty} \int_{1}^{t} \frac{dx}{10 - 2x + x^2} = \lim_{t \to \infty} \frac{1}{3} \tan^{-1} \left( \frac{t - 1}{3} \right) - 0 = \frac{1}{3} \cdot \frac{\pi}{2}.$

(4) Consider the triangular region in the $xy$-plane that is bounded by the lines $x = 0$, $y = H$ and $x = \frac{R}{H} y$. If this region is rotated about the $y$-axis then we get an upside-down cone with height $H$ and a circular base (at the top) with radius $R$. Being upside-down does not affect the volume of the cone. If we use the Method of Disks then the volume of the cone is

$$\int_{0}^{H} \pi \left( \frac{R}{H} y \right)^2 \, dy = \pi \frac{R^2}{H^2} \int_{0}^{H} y^2 \, dy = \pi \frac{R^2}{H^2} \frac{H^3}{3} = \frac{1}{3} \pi R^2 H.$$ 

(5) We use the same geometric setup that we had in problem (4). Note that the line $x = \frac{R}{H} y$ is the same as the line $y = \frac{H}{R} x$. If we use the Method of Shells then the volume of the cone is

$$\int_{0}^{R} 2\pi x \left( H - \frac{H}{R} x \right) \, dx = 2\pi H \int_{0}^{R} x - x^2 \frac{R}{H} \, dx = 2\pi H \left( \frac{R^2}{2} - \frac{R^3}{3R} \right) = \frac{1}{3} \pi R^2 H.$$ 

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(6) Integration by parts gives
\[
\int \sinh^n x \, dx = \int \sinh^{n-1} x \sinh x \, dx
\]
\[
= \sinh^{n-1} x \cosh x - \int ((n - 1) \sinh^{n-2} x \cosh x) \cosh x \, dx
\]
\[
= \sinh^{n-1} x \cosh x - (n - 1) \int \sinh^{n-2} x \cosh^2 x \, dx
\]
\[
= \sinh^{n-1} x \cosh x - (n - 1) \int \sinh^{n-2} x (1 + \sinh^2 x) \, dx
\]
\[
= \sinh^{n-1} x \cosh x - (n - 1) \int \sinh^{n-2} x \, dx - (n - 1) \int \sinh^n x \, dx.
\]
After moving the \((n - 1) \int \sinh^n x \, dx\) term to the other side of the equation, we obtain
\[
\int \sinh^n x \, dx + (n - 1) \int \sinh^n x \, dx = \sinh^{n-1} x \cosh x - (n - 1) \int \sinh^{n-2} x \, dx.
\]
This is the same as
\[
n \int \sinh^n x \, dx = \sinh^{n-1} x \cosh x - (n - 1) \int \sinh^{n-2} x \, dx.
\]
Now we divide both sides by \(n\) and get
\[
\int \sinh^n x \, dx = \frac{\sinh^{n-1} x \cosh x}{n} - \frac{n - 1}{n} \int \sinh^{n-2} x \, dx.
\]
(7) Two integrations by parts give
\[
\int x^3 \cos(\ln x) \, dx = \frac{x^4}{4} \cos(\ln x) - \int \frac{x^4}{4} \left( -\frac{1}{x} \sin(\ln x) \right) \, dx
\]
\[
= \frac{x^4}{4} \cos(\ln x) + \frac{1}{4} \int x^3 \sin(\ln x) \, dx
\]
\[
= \frac{x^4}{4} \cos(\ln x) + \frac{1}{4} \left( \frac{x^4}{4} \sin(\ln x) - \int \frac{x^4}{4} \left( \frac{1}{x} \cos(\ln x) \right) \, dx \right)
\]
\[
= \frac{x^4}{4} \cos(\ln x) + \frac{x^4}{16} \sin(\ln x) - \frac{1}{16} \int x^3 \cos(\ln x) \, dx.
\]
After moving \(\frac{1}{16} \int x^3 \cos(\ln x) \, dx\) to the other side of the equation, we obtain
\[
\int x^3 \cos(\ln x) \, dx + \frac{1}{16} \int x^3 \cos(\ln x) \, dx = \frac{x^4}{4} \cos(\ln x) + \frac{x^4}{16} \sin(\ln x) + C.
\]
This is the same as
\[ \frac{17}{16} \int x^3 \cos(\ln x) \, dx = \frac{x^4}{4} \cos(\ln x) + \frac{x^4}{16} \sin(\ln x) + C. \]

Now we divide both sides by \( \frac{17}{16} \) and get
\[ \int x^3 \cos(\ln x) \, dx = \frac{4x^4}{17} \cos(\ln x) + \frac{x^4}{17} \sin(\ln x) + C. \]

(8) If \( 1 \leq x < \infty \) then \(-x^6 \leq -x\). This implies \( 0 < e^{-x^6} \leq e^{-x} \) for \( 1 \leq x < \infty \). If we could show that \( \int_1^\infty e^{-x} \, dx \) converges, then the Comparison Theorem would imply that \( \int_1^\infty e^{-x^6} \, dx \) also converges. Showing that \( \int_1^\infty e^{-x} \, dx \) converges is easy, since
\[ \int_1^\infty e^{-x} \, dx = \lim_{t \to \infty} \int_1^t e^{-x} \, dx = \lim_{t \to \infty} -e^{-x} \bigg|_1^t = e^{-1}. \]

(9) Using integration by parts first and the trigonometric substitution \( x = \sin \theta, \sqrt{1-x^2} = \cos \theta, \, dx = \cos \theta \, d\theta \) afterwards, we get
\[ \int x \sin^{-1} x \, dx = \frac{x^2}{2} \sin^{-1} x - \frac{x^2}{2} \int \frac{1}{\sqrt{1-x^2}} \, dx = \frac{x^2}{2} \sin^{-1} x - \frac{1}{2} \int \frac{\sin^2 \theta \cos \theta \, d\theta}{\cos \theta} \]
\[ = \frac{x^2}{2} \sin^{-1} x - \frac{1}{2} \int \sin^2 \theta \, d\theta = \frac{x^2}{2} \sin^{-1} x - \frac{1}{4} \left[ \theta - \frac{1}{2} \sin(2\theta) \right] + C = \frac{x^2}{2} \sin^{-1} x - \frac{1}{4} \theta + \frac{1}{8} \sin(2\theta) + C \]
\[ = \frac{x^2}{2} \sin^{-1} x - \frac{1}{4} \theta + \frac{1}{4} \sin \theta \cos \theta + C \]
\[ = \frac{x^2}{2} \sin^{-1} x - \frac{1}{4} \sin^{-1} x + \frac{1}{4} x \sqrt{1-x^2} + C. \]

Average value \( = \frac{1}{1-0} \left[ \frac{1}{4} x \sqrt{1-x^2} \right]_0^1 = \frac{\pi}{8}. \)

(10) We have \( 4-x^2 = 1+x^2 \) when \( 2x^2 = 3 \), and this is \( x = \pm \sqrt{3/2} \). The Method of Washers tells us that the volume of the solid is
\[ \int_{-\sqrt{3/2}}^{\sqrt{3/2}} \pi (4-x^2)^2 - \pi (1+x^2)^2 \, dx = \pi \int_{-\sqrt{3/2}}^{\sqrt{3/2}} 16 - 1 - 8x^2 - 2x^2 \, dx \]
\[ = \pi \int_{-\sqrt{3/2}}^{\sqrt{3/2}} 15 - 10x^2 \, dx = \pi \left[ 15x - \frac{10x^3}{3} \right]_{-\sqrt{3/2}}^{\sqrt{3/2}} \]
\[ = 20\pi \sqrt{3/2}. \]