Solutions to Review Sheet for the Last Part of the Syllabus, Math 151

This is a review sheet for the sections in the syllabus that were not covered in the previous review sheets. When you study for the final exam, you should look at this review sheet and the review sheets for Exam 1 and Exam 2.

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(1) Consider the partition $a = x_0 < x_1 < x_2 < x_3 < x_4 = b$ where $x_0 = -4$, $x_1 = -1$, $x_2 = 5$, $x_3 = 10$, $x_4 = 20$. Assume $f(x) = -x^2$. Find the Riemann sum corresponding to this partition, this function $f(x)$ and the sample points $c_1 = -2$, $c_2 = 4$, $c_3 = 7$, $c_4 = 12$. What is the norm of this partition?

The Riemann sum is

$$(-(-2)^2)(-1-(-4)) + (-4^2)(5-(-1)) + (-7^2)(10-5) + (-12^2)(20-10).$$

The norm is the maximum of the numbers $-1-(-4)$, $5-(-1)$, $10-5$, $20-10$. Therefore, the norm is 10.

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(2) Evaluate $\displaystyle\int e^{5x}\,dx$, $\displaystyle\int e^{3x}\left(e^{-7x}+e^{2x}\right)dx$ and $\displaystyle\int \frac{e^{4x}-e^{5x}}{e^{2x}}\,dx$.

The indefinite integrals are

$$\int e^{5x}\,dx = \frac{1}{5}e^{5x} + C,$$

$$\int e^{3x}\left(e^{-7x}+e^{2x}\right)dx = \int e^{-4x} + e^{5x}\,dx = -\frac{1}{4}e^{-4x} + \frac{1}{5}e^{5x} + C,$$

$$\int \frac{e^{4x}-e^{5x}}{e^{2x}}\,dx = \int e^{2x} - e^{3x}\,dx = \frac{1}{2}e^{2x} - \frac{1}{3}e^{3x} + C.$$

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(3) Evaluate $\displaystyle\int (\sqrt{x}+x^2)(x^3-x^{-1/2})\,dx$ and $\displaystyle\int \frac{\sqrt{x}+x^4}{x^{3/2}}\,dx$.

Tne indefinite integrals are

$$\int (\sqrt{x}+x^2)(x^3-x^{-1/2})\,dx = \int x^{7/2} - 1 + x^5 - x^{3/2}\,dx = \frac{2x^{9/2}}{9} - x + \frac{x^6}{6} - \frac{2x^{5/2}}{5} + C,$$

$$\int \frac{\sqrt{x}+x^4}{x^{3/2}}\,dx = \int \frac{1}{x} + x^{5/2}\,dx = \ln|x| + \frac{2}{7}x^{7/2} + C.$$

Here is a question for you: Can we replace $\ln|x|$ by $\ln x$ in the second answer? Note that $\sqrt{x}$ is not defined when $x < 0$.

(4) Evaluate $\displaystyle\int \tan^2 x\,dx$ and $\displaystyle\int \cot^2 x\,dx$.

Trigonometric identities give us

$$\int \tan^2 x\,dx = \int \sec^2 x - 1\,dx = \tan x - x + C,$$

$$\int \cot^2 x\,dx = \int \csc^2 x - 1\,dx = -\cot x - x + C.$$

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(5) Evaluate $\displaystyle\int_{-e^3}^{-e^2} \frac{dx}{x}$ and $\displaystyle\int_{\pi/6}^{\pi/3} \sec x \tan x\,dx$.

The definite integrals are

$$\int_{-e^3}^{-e^2} \frac{dx}{x} = \ln|x|\Big|_{-e^3}^{-e^2} = \ln|-e^2| - \ln|-e^3| = \ln(e^2) - \ln(e^3) = 2\ln e - 3\ln e = 2 - 3 = -1,$$

$$\int_{\pi/6}^{\pi/3} \sec x \tan x\,dx = \sec x\Big|_{\pi/6}^{\pi/3} = \sec(\pi/3) - \sec(\pi/6) = 2 - \frac{2}{\sqrt{3}}.$$

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(6) Assume that $f(x)$ is a continuous function defined on the interval $[1, 12]$ such that $\displaystyle\int_1^7 f(x)\,dx = -3$, $\displaystyle\int_5^{12} f(x)\,dx = 4$, $\displaystyle\int_5^7 f(x)\,dx = -8$. Evaluate $\displaystyle\int_1^{12} f(x)\,dx$.

Using the basic formula $\displaystyle\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$, we see

$$\begin{aligned} -3 = \int_1^7 f(x)\,dx &= \int_1^5 f(x)\,dx + \int_5^7 f(x)\,dx \\ &= \int_1^5 f(x)\,dx + (-8), \quad \text{hence} \quad \int_1^5 f(x)\,dx = 5, \end{aligned}$$

$$\begin{aligned} 4 = \int_5^{12} f(x)\,dx &= \int_5^7 f(x)\,dx + \int_7^{12} f(x)\,dx \\ &= -8 + \int_7^{12} f(x)\,dx, \quad \text{hence} \quad \int_7^{12} f(x)\,dx = 12. \end{aligned}$$

Now we obtain

$$\int_1^{12} f(x)\,dx = \int_1^5 f(x)\,dx + \int_5^7 f(x)\,dx + \int_7^{12} f(x)\,dx = 5 + (-8) + 12 = 9.$$

(7) Evaluate $\dfrac{d}{dx}\displaystyle\int_2^{x^2} t^2 e^{t^2}\,dt$ and $\dfrac{d}{dx}\displaystyle\int_x^5 t(1+t^6)^{1/2}\,dt$.

We will use the function $A(x) = \displaystyle\int_2^x t^2 e^{t^2}\,dt$. Part II of the Fundamental Theorem of Calculus gives us the formula $A'(x) = x^2 e^{x^2}$. Now

$$\frac{d}{dx}\int_2^{x^2} t^2 e^{t^2}\,dt = \frac{d}{dx}\big(A(x^2)\big) = A'(x^2)2x = (x^2)^2 e^{(x^2)^2} 2x = x^4 e^{x^4} 2x.$$

The formula $\displaystyle\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$ and Part II of the Fundamental Theorem of Calculus lead to

$$\begin{aligned}\frac{d}{dx}\int_x^5 t(1+t^6)^{1/2}\,dt &= \frac{d}{dx}\left(-\int_5^x t(1+t^6)^{1/2}\,dt\right)\\ &= -\frac{d}{dx}\int_5^x t(1+t^6)^{1/2}\,dt = -x(1+x^6)^{1/2}.\end{aligned}$$

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(8) A particle has velocity $v(t) = t^3 - 5t^2 + 6t$ feet per second, where t is time measured in seconds. Find the displacement of the particle over the time interval $[1, 5]$. Find the total distance traveled over the time interval $[1, 5]$.

The displacement of the particle over the time interval $[1, 5]$ is

$$\int_1^5 v(t)\,dt = \int_1^5 t^3 - 5t^2 + 6t\,dt = \left(\frac{t^4}{4} - \frac{5t^3}{3} + 3t^2\right)\Bigg|_1^5.$$

Since $v(t) = t^3 - 5t^2 + 6t = t(t-2)(t-3)$, we have

$$v(t) \geq 0 \text{ when } 1 \leq t \leq 2,$$

$$v(t) \leq 0 \text{ when } 2 \leq t \leq 3,$$

$$v(t) \geq 0 \text{ when } 3 \leq t \leq 5.$$

This implies

$$|v(t)| = v(t) \text{ when } 1 \leq t \leq 2,$$

$$|v(t)| = -v(t) \text{ when } 2 \leq t \leq 3,$$

$$|v(t)| = v(t) \text{ when } 3 \leq t \leq 5.$$

Therefore, the total distance traveled over the time interval $[1, 5]$ is

$$
\begin{aligned}
\int_1^5 |v(t)|\,dt &= \int_1^2 |v(t)|\,dt + \int_2^3 |v(t)|\,dt + \int_3^5 |v(t)|\,dt \\
&= \int_1^2 v(t)\,dt + \int_2^3 -v(t)\,dt + \int_3^5 v(t)\,dt \\
&= \int_1^2 v(t)\,dt - \int_2^3 v(t)\,dt + \int_3^5 v(t)\,dt \\
&= \int_1^2 t^3 - 5t^2 + 6t\,dt - \int_2^3 t^3 - 5t^2 + 6t\,dt + \int_3^5 t^3 - 5t^2 + 6t\,dt \\
&= \left(\frac{t^4}{4} - \frac{5t^3}{3} + 3t^2\right)\Bigg|_1^2 - \left(\frac{t^4}{4} - \frac{5t^3}{3} + 3t^2\right)\Bigg|_2^3 + \left(\frac{t^4}{4} - \frac{5t^3}{3} + 3t^2\right)\Bigg|_3^5
\end{aligned}
$$

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(9) Evaluate these integrals:

$$
\int \tan x\,dx \qquad \int \cot x\,dx \qquad \int_1^2 x\sqrt{3x+2}\,dx \qquad \int x^2\sin(5x^3+7)\,dx
$$

$$
\int (5+\cos x)^3 \sin x\,dx \qquad \int \tan^4 x \sec^2 x\,dx \qquad \int \frac{x^3\,dx}{x^4+2} \qquad \int \frac{dx}{\sqrt{36-25x^2}}
$$

$$
\int \frac{x^3\,dx}{\sqrt{1-x^8}} \qquad \int \frac{x^4\,dx}{1+x^{10}} \qquad \int \frac{dx}{x^2+4x+5}
$$

The substitution $u = \sec x$, $du = \tan x \sec x\,dx$ gives

$$
\int \tan x\,dx = \int \frac{\tan x \sec x\,dx}{\sec x} = \int \frac{du}{u} = \ln|u| + C = \ln|\sec x| + C.
$$

The substitution $u = \sin x$, $du = \cos x\,dx$ gives

$$
\int \cot x\,dx = \int \frac{\cos x\,dx}{\sin x} = \int \frac{du}{u} = \ln|u| + C = \ln|\sin x| + C.
$$

The substitution $u = 3x+2$, $du = 3\,dx$, $x = \dfrac{u-2}{3}$, $dx = \dfrac{du}{3}$ gives

$$
\int_1^2 x\sqrt{3x+2}\,dx = \int_5^8 \frac{u-2}{3}\sqrt{u}\,\frac{du}{3} = \frac{1}{9}\int_5^8 u^{3/2} - 2u^{1/2}\,du = \frac{1}{9}\left(\frac{2u^{5/2}}{5} - \frac{4u^{3/2}}{3}\right)\Bigg|_5^8,
$$

where we used the following fact: The limits of integration $x=1$, $x=2$ correspond to the limits of integration $u = 3\cdot 1 + 2 = 5$, $u = 3\cdot 2 + 2 = 8$.

The substitution $u = 5x^3 + 7$, $du = 15x^2\,dx$, $\dfrac{du}{15} = x^2\,dx$ gives

$$\int x^2 \sin(5x^3+7)\,dx = \int \frac{\sin u\,du}{15} = -\frac{\cos u}{15} + C = -\frac{\cos(5x^3+7)}{15} + C.$$

The substitution $u = 5 + \cos x$, $du = -\sin x\,dx$, $-du = \sin x\,dx$ gives

$$\int (5+\cos x)^3 \sin x\,dx = -\int u^3\,du = -\frac{u^4}{4} + C = -\frac{(5+\cos x)^4}{4} + C.$$

The substitution $u = \tan x$, $du = \sec^2 x\,dx$ gives

$$\int \tan^4 x \sec^2 x\,dx = \int u^4\,du = \frac{u^5}{5} + C = \frac{\tan^5 x}{5} + C.$$

The substitution $u = x^4 + 2$, $du = 4x^3\,dx$, $\dfrac{1}{4}du = x^3\,dx$ gives

$$\int \frac{x^3\,dx}{x^4+2} = \frac{1}{4}\int \frac{du}{u} = \frac{1}{4}\ln|u| + C = \frac{1}{4}\ln|x^4+2| + C = \frac{1}{4}\ln(x^4+2) + C,$$

where we used the fact $x^4 + 2 > 0$, which implies $|x^4+2| = x^4+2$.
The substitution $u = (5/6)x$, $du = (5/6)dx$, $(6/5)du = dx$ gives

$$\begin{aligned}\int \frac{dx}{\sqrt{36-25x^2}} &= \int \frac{dx}{\sqrt{36(1-(25/36)x^2)}} = \int \frac{dx}{\sqrt{36}\sqrt{1-(25/36)x^2}} \\ &= \frac{1}{6}\int \frac{dx}{\sqrt{1-((5/6)x)^2}} = \frac{1}{6}\cdot\frac{6}{5}\int \frac{du}{\sqrt{1-u^2}} \\ &= \frac{1}{5}\sin^{-1} u + C = \frac{1}{5}\sin^{-1}((5/6)x) + C.\end{aligned}$$

The substitution $u = x^4$, $du = 4x^3\,dx$, $\dfrac{1}{4}du = x^3\,dx$ gives

$$\int \frac{x^3\,dx}{\sqrt{1-x^8}} = \frac{1}{4}\int \frac{du}{\sqrt{1-u^2}} = \frac{1}{4}\sin^{-1} u + C = \frac{1}{4}\sin^{-1}(x^4) + C.$$

The substitution $u = x^5$, $du = 5x^4\,dx$, $\dfrac{1}{5}du = x^4\,dx$ gives

$$\int \frac{x^4\,dx}{1+x^{10}} = \frac{1}{5}\int \frac{du}{1+u^2} = \frac{1}{5}\tan^{-1} u + C = \frac{1}{5}\tan^{-1}(x^5) + C.$$

Completing the square, we obtain $x^2+4x+5 = (x+2)^2+1$. Now the substitution $u = x+2$, $du = dx$ gives

$$\int \frac{dx}{x^2+4x+5} = \int \frac{dx}{(x+2)^2+1} = \int \frac{du}{u^2+1} = \tan^{-1} u + C = \tan^{-1}(x+2) + C.$$

(10) A bacterial population in a petri dish triples in size every 5 days. How long does it take for this population to double in size?

The size of the population at time t (in days) is $P(t) = P_0 e^{kt}$. Our assumptions say $P(5) = 3P(0)$, which is $P_0 e^{k\cdot 5} = 3P_0 e^{k\cdot 0} = 3P_0$. After dividing by P_0, we get $e^{5k} = 3$. After taking the logarithm of both sides, we obtain $5k = \ln 3$, which is $k = \dfrac{\ln 3}{5}$. Let T be the number of days it takes for the population to double. We know $P(T) = 2P(0)$, which is $P_0 e^{kT} = 2P_0 e^{k\cdot 0} = 2P_0$. After dividing by P_0, we get $e^{kT} = 2$, hence $kT = \ln 2$ and $T = \dfrac{\ln 2}{k}$. Substituting the formula $k = \dfrac{\ln 3}{5}$, which we derived earlier, we get $T = \dfrac{5\ln 2}{\ln 3}$.

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(11) We start out with a 7 microgram sample of a certain radioactive isotope. After 8 days have passed, we find that 2 micrograms of the sample have decayed. What is the half-life of this isotope?

The mass of the isotope at time t (in days) is $P(t) = P_0 e^{kt}$ (in micrograms). Our assumptions say $P(0) = 7$ and $P(8) = 7 - 2 = 5$. We know $P(0) = P_0 e^{k\cdot 0} = P_0$. All this implies

$$P_0 = P(0) = 7, \text{ hence } 5 = P(8) = P_0 e^{k\cdot 8} = 7e^{8k}, \text{ hence } e^{8k} = 5/7.$$

From $e^{8k} = 5/7$ we get $8k = \ln(5/7)$, which is $k = \dfrac{\ln(5/7)}{8}$. Let T be the half-life of the isotope. We know $P(T) = (1/2)P(0)$, which is $P_0 e^{kT} = (1/2)P_0 e^{k\cdot 0} = (1/2)P_0$. After dividing by P_0, we get $e^{kT} = 1/2$, hence $kT = \ln(1/2) = -\ln 2$ and $T = -\dfrac{\ln 2}{k}$. Substituting the formula $k = \dfrac{\ln(5/7)}{8}$, which we derived earlier, we get $T = \dfrac{-8\ln 2}{\ln(5/7)} = \dfrac{-8\ln 2}{-\ln(7/5)} = \dfrac{8\ln 2}{\ln(7/5)}$.

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(12) Find the area of the region bounded by $y = x^2$ and $y = 7x - 10$.

The curves $y = x^2$ and $y = 7x - 10$ intersect when $x^2 = 7x - 10$, which is $x^2 - 7x + 10 = 0$. After solving $x^2 - 7x + 10 = 0$, we get $x = 2$ and $x = 5$. This tells us that the interval of integration is $[2, 5]$. Since $y = x^2$ is concave up and $y = 7x - 10$ is a straight line, we know that $y = 7x - 10$ is higher than $y = x^2$ on $[2, 5]$. The required area is

$$\int_2^5 (7x - 10) - x^2\,dx = \left(\frac{7x^2}{2} - 10x - \frac{x^3}{3}\right)\Bigg|_2^5.$$