Solutions to Review Sheet for Exam 2, Math 151

(1) Consider the function $f(x) = 3x^5 - 10x^3 + 5x$. Find the intervals where the function is increasing and the intervals where the function is decreasing. Find the intervals where the function is concave up and the intervals where the function is concave down. Find all relative maxima, all relative minima and all inflection points.

We see $f'(x) = 15x^4 - 30x^2 + 5 = 15(x^4 - 2x^2 + 1/3) = 15[(x^2-1)^2 - 2/3]$. Therefore, the critical numbers are $\pm\sqrt{1 \pm \sqrt{2/3}}$. The function $f(x)$ is

$$\text{increasing on } \left(-\infty, -\sqrt{1+\sqrt{2/3}}\right)\ ,\ \left(-\sqrt{1-\sqrt{2/3}}, \sqrt{1-\sqrt{2/3}}\right)\ ,\ \left(\sqrt{1+\sqrt{2/3}}, \infty\right)$$

$$\text{decreasing on } \left(-\sqrt{1+\sqrt{2/3}}, -\sqrt{1-\sqrt{2/3}}\right)\ ,\quad \left(\sqrt{1-\sqrt{2/3}}, \sqrt{1+\sqrt{2/3}}\right)\ .$$

There are relative maxima at $-\sqrt{1+\sqrt{2/3}}$ and $\sqrt{1-\sqrt{2/3}}$.
There are relative minima at $-\sqrt{1-\sqrt{2/3}}$ and $\sqrt{1+\sqrt{2/3}}$.
We also see $f''(x) = 15(4x^3 - 4x) = 60x(x^2-1)$. The function $f(x)$ is

$$\text{concave up on } (-1,0)\,,\ (1,\infty)\ ,\qquad \text{concave down on } (-\infty,-1)\,,\ (0,1)$$

There are inflection points at -1, 0, 1.

(2) Now consider the function $f(x) = 3x^5 - 10x^3 - 5x$. Find the intervals where the function is increasing and the intervals where the function is decreasing. Find the intervals where the function is concave up and the intervals where the function is concave down. Find all relative maxima, all relative minima and all inflection points.

We see $f'(x) = 15x^4 - 30x^2 - 5 = 15(x^4 - 2x^2 - 1/3) = 15[(x^2-1)^2 - 4/3]$. Therefore, the critical numbers are $\pm\sqrt{1 + \sqrt{4/3}}$. The function $f(x)$ is

$$\text{increasing on } \left(-\infty, -\sqrt{1+\sqrt{4/3}}\right)\ ,\ \left(\sqrt{1+\sqrt{4/3}}, \infty\right)$$

$$\text{decreasing on } \left(-\sqrt{1+\sqrt{4/3}}, \sqrt{1+\sqrt{4/3}}\right)$$

There is a relative maximum at $-\sqrt{1+\sqrt{4/3}}$. There is a relative minimum at $\sqrt{1+\sqrt{4/3}}$. The second derivative of $f(x)$ is the same as the second derivative in Problem (1). Therefore, we still have the following: The function $f(x)$ is concave up on $(-1,0)$, $(1,\infty)$, concave down on $(-\infty,-1)$, $(0,1)$. There are inflection points at -1, 0, 1.

(3) A road running North-South intersects a road running East-West. A green car is traveling south at 70 miles per hour on the North-South road and it is heading toward the

intersection. A blue car is traveling east on the East-West road and it is heading away from the intersection. There is a moment when the separation between the two cars is 50 miles, the separation between the two cars is decreasing at the rate of 10 miles per hour, and the blue car is 40 miles away from the intersection. What is the speed of the blue car at that moment?

We will use miles and hours as units. Let x be the distance from the intersection to the blue car. Let y be the distance from the intersection to the green car. Let z be the distance between the two cars. If we take $\frac{d}{dt}$ of the Pythagorean equation $x^2 + y^2 = z^2$ then we obtain $2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2z\frac{dz}{dt}$, which simplifies to the key equation

$$x\frac{dx}{dt} + y\frac{dy}{dt} = z\frac{dz}{dt}.$$

At the moment in question, $\frac{dy}{dt} = -70$, $\frac{dz}{dt} = -10$, $x = 40$, $z = 50$. The Pythagorean Theorem implies $y = 30$ at that moment. When we substitute all this into the key equation and solve, we get $\frac{dx}{dt} = 40$. This is the speed of the blue car at that moment, in miles per hour.

(4) Find $\lim\limits_{x\to 0} \dfrac{3x - \sin(3x)}{5x - \sin(5x)}$ using L'Hôpital's Rule several times.

Three uses of L'Hôpital's Rule give us

$$\begin{aligned}\lim_{x\to 0} \frac{3x - \sin(3x)}{5x - \sin(5x)} &= \lim_{x\to 0} \frac{3 - 3\cos(3x)}{5 - 5\cos(5x)} = \frac{3}{5}\lim_{x\to 0} \frac{1 - \cos(3x)}{1 - \cos(5x)} \\ &= \frac{3}{5}\lim_{x\to 0} \frac{3\sin(3x)}{5\sin(5x)} = \frac{9}{25}\lim_{x\to 0} \frac{\sin(3x)}{\sin(5x)} \\ &= \frac{9}{25}\lim_{x\to 0} \frac{3\cos(3x)}{5\cos(5x)} = \frac{27}{125}\cdot\frac{\cos(0)}{\cos(0)} = \frac{27}{125}\cdot\frac{1}{1} = \frac{27}{125}\ .\end{aligned}$$

(5) A cardboard rectangle measures 30 inches by 10 inches. A square measuring x inches by x inches is cut out of each of the corners of this rectangle. What is left over is a twelve-sided piece of cardboard that can be folded into an open box that has a rectangular base measuring $30 - 2x$ inches by $10 - 2x$ inches and has a height of x inches. What value of x will maximize the volume of this open box?

Since the four square cutouts cannot overlap, we must have $0 \le x \le 5$. We are maximizing the function $f(x) = (30 - 2x)(10 - 2x)x = 300x - 80x^2 + 4x^3$ over the interval $[0, 5]$. To find the critical points, we solve $0 = f'(x) = 300 - 160x + 12x^2$. The quadratic formula gives $x = \dfrac{20 \pm 5\sqrt{7}}{3}$. Since we are only interested in critical points in the interval $(0, 5)$, we are only concerned with the solution $x = \dfrac{20 - 5\sqrt{7}}{3}$ of $0 = f'(x)$. When $x = 0$, $x = 5$

and $x = \dfrac{20-5\sqrt{7}}{3}$ are substituted into $f(x)$, only $x = \dfrac{20-5\sqrt{7}}{3}$ gives a positive $f(x)$. Therefore, the maximum volume occurs at $x = \dfrac{20-5\sqrt{7}}{3}$.

(6) Consider the function $f(x) = \dfrac{\sqrt{(5x^2+1)(x^2-4x+4)}}{x^2-2x}$. Find all horizontal and vertical asymptotes of this function.

We will use the simplification

$$\begin{aligned}\frac{\sqrt{(5x^2+1)(x^2-4x+4)}}{x^2-2x} &= \frac{\sqrt{(5x^2+1)(x-2)^2}}{x(x-2)} = \frac{\sqrt{5x^2+1}\sqrt{(x-2)^2}}{x(x-2)} \\ &= \frac{\sqrt{5x^2+1}}{x}\cdot\frac{|x-2|}{x-2}.\end{aligned}$$

Since $\frac{|x-2|}{x-2} = \pm 1$, the only vertical asysmptote is $x = 0$. Now we will find the horizontal asymptotes. If $x \to \infty$ then eventually $x > 0$ and $x - 2 > 0$, which implies $x = \sqrt{x^2}$ and $|x-2| = x-2$. This gives

$$\begin{aligned}\lim_{x\to\infty}\frac{\sqrt{(5x^2+1)(x^2-4x+4)}}{x^2-2x} &= \lim_{x\to\infty}\frac{\sqrt{5x^2+1}}{x}\cdot\frac{|x-2|}{x-2} = \lim_{x\to\infty}\frac{\sqrt{5x^2+1}}{\sqrt{x^2}}\cdot\frac{x-2}{x-2} \\ &= \lim_{x\to\infty}\sqrt{\frac{5x^2+1}{x^2}} = \lim_{x\to\infty}\sqrt{5+\frac{1}{x^2}} = \sqrt{5}.\end{aligned}$$

Similarly: If $x \to -\infty$ then eventually $x < 0$ and $x - 2 < 0$, which implies $x = -\sqrt{x^2}$ and $|x-2| = -(x-2)$. This gives

$$\begin{aligned}\lim_{x\to-\infty}\frac{\sqrt{(5x^2+1)(x^2-4x+4)}}{x^2-2x} &= \lim_{x\to-\infty}\frac{\sqrt{5x^2+1}}{x}\cdot\frac{|x-2|}{x-2} \\ &= \lim_{x\to-\infty}\frac{\sqrt{5x^2+1}}{-\sqrt{x^2}}\cdot\frac{-(x-2)}{x-2} \\ &= \lim_{x\to-\infty}\frac{\sqrt{5x^2+1}}{\sqrt{x^2}}\cdot\frac{x-2}{x-2} \\ &= \lim_{x\to-\infty}\sqrt{\frac{5x^2+1}{x^2}} = \lim_{x\to-\infty}\sqrt{5+\frac{1}{x^2}} = \sqrt{5}.\end{aligned}$$

Thus, the only horizontal asymptote is $y = \sqrt{5}$.

(7) Consider the function $f(x) = \begin{cases} x^2 \ln x & \text{if } x > 0, \\ 0 & \text{if } x = 0. \end{cases}$

This function is defined on $[0, \infty)$. Show that this function is continuous at $x = 0$. Find the intervals where the function is increasing and the intervals where the function is decreasing.

Find the intervals where the function is concave up and the intervals where the function is concave down. Find the absolute minimum of the function and all inflection points of the function.

Continuity at 0 follows from the computation

$$\lim_{x\to 0^+} x^2 \ln x = \lim_{x\to 0^+} \frac{\ln x}{x^{-2}} = \lim_{x\to 0^+} \frac{\frac{d}{dx}\ln x}{\frac{d}{dx}x^{-2}} = \lim_{x\to 0^+} \frac{x^{-1}}{(-2)x^{-3}} = \lim_{x\to 0^+} \frac{x^2}{-2} = 0 \ ,$$

which used L'Hôpital's Rule. The first derivative of the function is

$$x + 2x\ln x = x(1 + 2\ln x),$$

which is positive on $(e^{-1/2}, \infty)$ and negative on $(0, e^{-1/2})$. This means that the function is decreasing on $(0, e^{-1/2})$ and increasing on $(e^{-1/2}, \infty)$. This gives us an absolute minimum at $x = e^{-1/2}$. The second derivative of the function is $3 + 2\ln x$, which is positive on $(e^{-3/2}, \infty)$ and negative on $(0, e^{-3/2})$. This means that the function is concave down on $(0, e^{-3/2})$ and concave up on $(e^{-3/2}, \infty)$. This gives us an inflection point at $x = e^{-3/2}$.

(8) Find the two points on the curve $y = x^2 - 2$ that are closest to the point $(0, 10)$.

We have to find the points (x, y) on the curve $y = x^2 - 2$ that minimize the distance $\sqrt{(x-0)^2 + (y-10)^2} = \sqrt{x^2 + (y-10)^2}$. These points are precisely the points (x, y) on the curve $y = x^2 - 2$ that minimize the distance squared, which is $x^2 + (y-10)^2$. Since the formula $x^2 = y + 2$ is valid on the curve $y = x^2 - 2$, we are actually minimizing $y + 2 + (y-10)^2$, and we are minimizing $f(y) = y + 2 + (y-10)^2$ for y in the interval $[-2, \infty)$. We know that $[-2, \infty)$ is the correct interval because the equation $y = x^2 - 2$ only makes sense when $y \geq -2$. The critical point is the solution of the equation $0 = f'(y) = 1 + 2(y - 10)$. This solution is $y = 19/2$. Since $f''(y) = 2 > 0$, the function f is concave up everywhere. This means that the critical point $y = 19/2$ is the point where we have an absolule minimum. Since we are looking at the curve $y = x^2 - 2$, the x values corresponding to $y = 19/2$ are $x = \pm\sqrt{23/2}$. Therefore, the points on the curve $y = x^2 - 2$ closest to $(0, 10)$ are the two points $(\pm\sqrt{23/2}, 19/2)$.

(9) Consider the function $f(x) = \dfrac{x^3}{x^2 - 1}$. Find the intervals where the function is increasing and the intervals where the function is decreasing. Find the intervals where the function is concave up and the intervals where the function is concave down. Find all relative minima and all relative maxima of the function. Find all inflection points of the function. Find the vertical asymptotes of the function.

We have

$$f'(x) = \frac{3x^2(x^2-1) - x^3(2x)}{(x^2-1)^2} = \frac{x^4 - 3x^2}{(x^2-1)^2} = \frac{x^2(x^2-3)}{(x^2-1)^2}.$$

This implies

$$f(x) \text{ is increasing on } (-\infty, -\sqrt{3})\ ,\ (\sqrt{3}, \infty)$$

and

$$f(x) \text{ is decreasing on } (-\sqrt{3},-1)\,,\,(-1,1)\,,\,(1,\sqrt{3}).$$

Note that $f(x)$ is NOT decreasing on $(-\sqrt{3},\sqrt{3})$, because we have vertical asymptotes at the intermediate points $x=-1$ and $x=1$ (where one might say that $f(x)$ "jumps up" from $-\infty$ to ∞). There is a relative maximum at $x=-\sqrt{3}$. There is a relative minimum at $x=\sqrt{3}$. We also have

$$\begin{aligned} f''(x) &= \frac{(4x^3-6x)(x^2-1)^2-(x^4-3x^2)2(x^2-1)2x}{(x^2-1)^4} \\ &= \frac{(4x^3-6x)(x^2-1)-(x^4-3x^2)4x}{(x^2-1)^3} = \frac{2x^3+6x}{(x^2-1)^3} = \frac{x(2x^2+6)}{(x^2-1)^3}, \end{aligned}$$

which implies

$$f(x) \text{ is concave down on } (-\infty,-1)\,,\,(0,1)$$

and

$$f(x) \text{ is concave up on } (-1,0)\,,\,(1,\infty).$$

(10) We wish to use Newton's Method to approximate $2^{1/5}$. What function $f(x)$ would be convenient in this case? Assume that the successive approximations are $x_0, x_1, x_2, \ldots$. Also assume $x_0=1$. Find x_1 and x_2.

We use Newton's Method to approximate the solution of $f(x)=0$ where $f(x)=x^5-2$. Newton's Method is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^5-2}{5x_n^4} = \frac{4}{5}x_n + \frac{2}{5}\cdot\frac{1}{x_n^4}\ .$$

Since $x_0=1$, we get $x_1=\frac{6}{5}$ and $x_2=\frac{24}{25}+\frac{2}{5}\cdot\frac{5^4}{6^4}$.

(11) When the radius of a sphere is measured, the measurement has a 1 percent error. What is the resulting percentage error when the volume of the sphere is computed using this inaccurate measurement of the radius?

We use the formula $V=(4/3)\pi R^3$, where R is the radius of the sphere, and V is its volume. Differentiating, we obtain $\dfrac{dV}{dR}=4\pi R^2$. This leads to the approximation $\dfrac{\Delta V}{\Delta R}\approx 4\pi R^2$, which can be rewritten $\Delta V \approx 4\pi R^2\Delta R$. The approximate equation $\Delta V\approx 4\pi R^2\Delta R$ and the exact equation $V=(4/3)\pi R^3$ combine to give us $\dfrac{\Delta V}{V}\approx\dfrac{4\pi R^2\Delta R}{(4/3)\pi R^3}$, which simplifies as $\dfrac{\Delta V}{V}\approx 3\dfrac{\Delta R}{R}$. Combining this with the assumption $\dfrac{\Delta R}{R}=1\%$, we get $\dfrac{\Delta V}{V}\approx 3\dfrac{\Delta R}{R}=3(1\%)=3\%$. Therefore, there is a 3 percent error in the computation of the volume.

(12) Evaluate $\dfrac{d}{dx}\left[\left(x^{\tan x}\right)^{\sec x}\right]$.

Note: $\left(x^{\tan x}\right)^{\sec x} = x^{\tan x \sec x}$. We will now compute $\dfrac{d}{dx}\ln\left(x^{\tan x \sec x}\right)$ in two different ways. First, we use $\frac{d}{dx}(\tan x \sec x) = \sec^3 x + \tan^2 x \sec x$ to find

$$\frac{d}{dx}\ln\left(x^{\tan x \sec x}\right) = \frac{d}{dx}(\tan x \sec x \ln x) = (\sec^3 x + \tan^2 x \sec x)\ln x + \frac{\tan x \sec x}{x}.$$

Then we use the formula $\dfrac{d}{dx}\ln(g(x)) = \dfrac{\frac{d}{dx}g(x)}{g(x)}$ to obtain

$$\frac{d}{dx}\ln\left(x^{\tan x \sec x}\right) = \frac{\frac{d}{dx}\left(x^{\tan x \sec x}\right)}{x^{\tan x \sec x}}.$$

When we equate these two ways of expressing $\dfrac{d}{dx}\ln\left(x^{\tan x \sec x}\right)$ we get

$$(\sec^3 x + \tan^2 x \sec x)\ln x + \frac{\tan x \sec x}{x} = \frac{\frac{d}{dx}\left(x^{\tan x \sec x}\right)}{x^{\tan x \sec x}}.$$

Solving for $\dfrac{d}{dx}\left(x^{\tan x \sec x}\right)$, we get

$$\frac{d}{dx}\left(x^{\tan x \sec x}\right) = \left(x^{\tan x \sec x}\right)\left((\sec^3 x + \tan^2 x \sec x)\ln x + \frac{\tan x \sec x}{x}\right).$$

(13) Evaluate $\dfrac{d}{dx}\left[\sec^{-1}\left(-5^{x^2+1}\right)\right]$.

The derivative is $\dfrac{-\ln(5)(2x)5^{x^2+1}}{\left|-5^{x^2+1}\right|\sqrt{\left(-5^{x^2+1}\right)^2-1}} = \dfrac{-\ln(5)(2x)5^{x^2+1}}{\left(5^{x^2+1}\right)\sqrt{5^{2x^2+2}-1}}$.

(14) Evaluate $\dfrac{d}{dx}\left[\tan^{-1}(x\sin^{-1}x)\right]$.

The derivative is $\dfrac{\frac{x}{\sqrt{1-x^2}} + \sin^{-1}x}{(x\sin^{-1}x)^2+1}$.

(15) Express $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ in terms of x and y when y is defined implicitly by $x^2+y^2=25$.

We will use the abbreviations y' and y'' for $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$, respectively. Implicit differentiation of $x^2+y^2=25$ gives $2x+2yy'=0$, which simplifies to $x+yy'=0$. Solving, we get $y'=-x/y$. When we differentiate $x+yy'=0$ implicitly, we get $1+y'y'+yy''=0$. Substituting $y'=-x/y$ into this last equation, we get $1+(-x/y)(-x/y)+yy''=0$. This is the same as $\dfrac{y^2+x^2}{y^2}+yy''=0$. Since $x^2+y^2=25$, we can rewrite this as $\dfrac{25}{y^2}+yy''=0$. Solving for y'', we get $y''=-\dfrac{25}{y^3}$.

(16) Evaluate $\lim_{x\to 0^+}(1+x)^{1/x}$ using L'Hôpital's Rule.

First, we use L'Hôpital's Rule to do the following simpler computation:

$$\lim_{x\to 0^+}\ln\left((1+x)^{1/x}\right)=\lim_{x\to 0^+}\frac{\ln(1+x)}{x}=\lim_{x\to 0^+}\frac{\frac{1}{1+x}}{1}=1\ .$$

Now we apply the exponential function to both sides of this equation and get

$$\lim_{x\to 0^+}(1+x)^{1/x}=e^1=e\ .$$

By the way, we used the continuity of the exponential function.

(17) Consider the function $f(x)=xe^{-x^4}$. Find the intervals where the function is increasing and the intervals where the function is decreasing. Find the intervals where the function is concave up and the intervals where the function is concave down.

We get

$$f'(x)=e^{-x^4}+x(-4x^3)e^{-x^4}=(1-4x^4)e^{-x^4}\ ,$$

$$f''(x)=-16x^3e^{-x^4}+(1-4x^4)(-4x^3)e^{-x^4}=(-16x^3-4x^3+16x^7)e^{-x^4}=4x^3(4x^4-5)e^{-x^4}.$$

The function $f(x)$ is increasing on $(-1/\sqrt{2},1/\sqrt{2})$ and decreasing on $(-\infty,-1/\sqrt{2})$ and $(1/\sqrt{2},\infty)$. The function $f(x)$ is concave down on $(-\infty,-(5/4)^{1/4})$ and $(0,(5/4)^{1/4})$. The function $f(x)$ is concave up on $(-(5/4)^{1/4},0)$ and $((5/4)^{1/4},\infty)$.

(18) Consider the function $f(x)=\dfrac{1}{x^2+2x+2}$. Find the intervals where the function is increasing and the intervals where the function is decreasing. Find the intervals where the function is concave up and the intervals where the function is concave down.

Since $f(x)=\dfrac{1}{(x+1)^2+1}$, we find

$$f'(x)=\frac{-2(x+1)}{[(x+1)^2+1]^2}\ ,$$

$$\begin{aligned} f''(x)&=\frac{-2[(x+1)^2+1]^2+2(x+1)2[(x+1)^2+1]2(x+1)}{[(x+1)^2+1]^4}\\ &=\frac{-2[(x+1)^2+1]+8(x+1)^2}{[(x+1)^2+1]^3}=\frac{6(x+1)^2-2}{[(x+1)^2+1]^3}\ . \end{aligned}$$

The function $f(x)$ is increasing on $(-\infty, -1)$ and decreasing on $(-1, \infty)$. The function $f(x)$ is concave down on $(-1 - 1/\sqrt{3}, -1 + 1/\sqrt{3})$. The function $f(x)$ is concave up on $(-\infty, -1 - 1/\sqrt{3})$ and $(-1 + 1/\sqrt{3}, \infty)$

(19) Consider a differentiable function $f(x)$ such that $f(2) = 7$ and $f'(x) > 3$ for all $x > 2$. What can you conclude about $f(5)$? Hint: MVT.

According to the Mean Value Theorem, there exists $c > 2$ such that $\dfrac{f(5) - f(2)}{5 - 2} = f'(c)$. Since $c > 2$, the hypotheses give $f'(c) > 3$. Now we know

$$\frac{f(5) - 7}{3} = \frac{f(5) - f(2)}{5 - 2} = f'(c) > 3.$$

Solving this inequality, we obtain $f(5) - 7 > 3 \cdot 3 = 9$, hence $f(5) > 9 + 7 = 16$.

(20) Find the area of the region in the xy plane that is bounded by the lines $x = 1$, $x = 3$, $y = 0$, $y = x$ in the following way: Approximate the area using the sum of the areas of n thin rectangles. Let n approach ∞.

For any positive integer n, the area of that region is approximated by the sum of the areas of the rectangles with base $[1+2(j-1)/n, 1+2j/n]$ and height $1+2j/n$, for $j = 1, 2, 3, \ldots, n$. This sum is the sum of the numbers $(2/n)(1+2j/n)$, for $j = 1, 2, 3, \ldots, n$. In other words, the area of that region is approximated by the sum

$$\begin{aligned}\sum_{j=1}^{n}(2/n)(1+2j/n) &= (2/n)\sum_{j=1}^{n}(1+2j/n) = (2/n)\left(\left(\sum_{j=1}^{n} 1\right) + \left(\sum_{j=1}^{n} 2j/n\right)\right) \\ &= (2/n)\left(n + (2/n)\left(\sum_{j=1}^{n} j\right)\right) = (2/n)\left(n + (2/n)\left(\frac{n(n+1)}{2}\right)\right) \\ &= \frac{2(2n+1)}{n}\ .\end{aligned}$$

When n approaches ∞, This sum approaches 4, which is the correct area of the region.