Solutions to Review Sheet for Exam 1, Math 151

These problems are presented in order to help you understand the material that is listed prior to the first exam in the syllabus. DO NOT assume that your first midterm exam will resemble this set of problems. The following 20 problems are not meant to be a sample exam. These problems are just a study aid. Since we have not covered L'Hôpital's Rule yet, this rule should not be used to answer any of these questions.

(1) Simplify $\sin^{-1}(\sin(7\pi/4))$ and $\sin^{-1}(\sin(5\pi/6))$.

Recall that the formula $\sin^{-1}(\sin(x)) = x$ is valid only when $-\pi/2 \le x \le \pi/2$.
$\sin(7\pi/4) = \sin(7\pi/4 - 2\pi) = \sin(-\pi/4)$ and $-\pi/2 < -\pi/4 < \pi/2$. Therefore,

$$\sin^{-1}(\sin(7\pi/4)) = \sin^{-1}(\sin(-\pi/4)) = -\pi/4.$$

$\sin(5\pi/6) = -\sin(5\pi/6 - \pi) = -\sin(-\pi/6) = \sin(\pi/6)$ and $-\pi/2 < \pi/6 < \pi/2$. Therefore,

$$\sin^{-1}(\sin(5\pi/6)) = \sin^{-1}(\sin(\pi/6)) = \pi/6.$$

(2) Assume x_0 has the properties $\sin(x_0) = 0.7$ and $\cos(x_0) < 0$. Evaluate $\sin(4x_0)$ and $\cos(4x_0)$.

We know $\cos(x_0) = \pm\sqrt{1-\sin^2(x_0)} = \pm\sqrt{1-(\sin(x_0))^2} = \pm\sqrt{1-(0.7)^2} = \pm\sqrt{0.51}$ and $\cos(x_0)$ is negative. Therefore, the correct choice of $\pm$ is $-$ and $\cos(x_0) = -\sqrt{0.51}$. Now

$$\sin(2x_0) = 2\sin(x_0)\cos(x_0) = 2(0.7)(-\sqrt{0.51}) = -(1.4)\sqrt{0.51}$$

and

$$\cos(2x_0) = \cos^2(x_0) - \sin^2(x_0) = 0.51 - (0.7)^2 = 0.02.$$

Then

$$\sin(4x_0) = 2\sin(2x_0)\cos(2x_0) = 2(-(1.4)\sqrt{0.51})(0.02) = -0.056\sqrt{0.51}$$

and

$$\begin{aligned}\cos(4x_0) &= \cos^2(2x_0) - \sin^2(2x_0) = (0.02)^2 - (-(1.4)\sqrt{0.51})^2 \\ &= 0.0004 - (1.96)(0.51) = -0.9992.\end{aligned}$$

(3) Simplify $\cot(\sin^{-1} x)$ and $\cos(\tan^{-1} x)$.

We know $(\cos(\sin^{-1} x))^2 = \cos^2(\sin^{-1} x) = 1 - \sin^2(\sin^{-1} x) = 1 - (\sin(\sin^{-1} x))^2 = 1 - x^2$, hence $\cos(\sin^{-1} x) = \pm\sqrt{1-x^2}$. Since $\sin^{-1} x$ is in the interval $[-\pi/2, \pi/2]$, we must have $\cos(\sin^{-1} x) \ge 0$, and this implies that the $\pm$ must be $+$. This says $\cos(\sin^{-1} x) = \sqrt{1-x^2}$. We also know $\sin(\sin^{-1} x) = x$. This implies

$$\cot(\sin^{-1} x) = \frac{\cos(\sin^{-1} x)}{\sin(\sin^{-1} x)} = \frac{\sqrt{1-x^2}}{x}.$$

We know $(\sec(\tan^{-1} x))^2 = \sec^2(\tan^{-1} x) = 1 + \tan^2(\tan^{-1} x) = 1 + (\tan(\tan^{-1} x))^2 = 1 + x^2$, hence $\sec(\tan^{-1} x) = \pm\sqrt{1+x^2}$. Since $\tan^{-1} x$ is in the interval $(-\pi/2, \pi/2)$, we must have $\sec(\tan^{-1} x) > 0$, and this implies that the $\pm$ must be $+$. This says $\sec(\tan^{-1} x) = \sqrt{1+x^2}$. This implies

$$\cos(\tan^{-1} x) = \frac{1}{\sec(\tan^{-1} x)} = \frac{1}{\sqrt{1+x^2}}.$$

(4) Find all solutions of the equation $\dfrac{\left(e^{6+x^2/2}\right)^2}{e^{7x}} = 1$.

Since $(e^t)^2 = e^{2t}$, this can rewritten as $\dfrac{e^{2(6+x^2/2)}}{e^{7x}} = 1$. This equation is equivalent to $e^{2(6+x^2/2)} = e^{7x}$. Since $e^u = e^v$ implies $u = v$, we now know $2(6 + x^2/2) = 7x$, which is $x^2 - 7x + 12 = 0$. The quadratic equation gives us the solutions $x = 3$ and $x = 4$.

(5) Solve for x in the equation $\log_{27} x = 1/3$.

From $\log_{27} x = 1/3$ we get $27^{\log_{27} x} = 27^{1/3} = 3$. Since $27^{\log_{27} x} = x$, we get $x = 3$.

(6) The position of a particle at time t seconds is $s = t^2 + 1/t$ meters. Find the average velocity over the time interval $[1, 5]$.

The average velocity over the time interval $[1, 5]$ is

$$\frac{(5^2 + 1/5) - (1^2 + 1/1)}{5 - 1} = 5.8 \text{ meters/second.}$$

(7) Find $\displaystyle\lim_{x\to 5^+} \frac{x-5}{|x-5|}$ and $\displaystyle\lim_{x\to 5^-} \frac{x-5}{|x-5|}$.

If x approaches 5 from the right then $x > 5$ and $x - 5 > 0$, which implies $|x-5| = x - 5$. This gives

$$\lim_{x\to 5^+} \frac{x-5}{|x-5|} = \lim_{x\to 5^+} \frac{x-5}{x-5} = \lim_{x\to 5^+} 1 = 1.$$

If x approaches 5 from the left then $x < 5$ and $x - 5 < 0$, which implies $|x-5| = -(x-5)$. This gives

$$\lim_{x\to 5^-} \frac{x-5}{|x-5|} = \lim_{x\to 5^-} \frac{x-5}{-(x-5)} = \lim_{x\to 5^-} -1 = -1.$$

(8) Find $\displaystyle\lim_{x\to 4^+} \frac{10-x^2}{(x-4)^5}$ and $\displaystyle\lim_{x\to 4^-} \frac{10-x^2}{(x-4)^5}$.

As x approaches 4 from the right, $x - 4$ approaches 0 and remains positive. This implies that $(x-4)^5$ approaches 0 and remains positive as x approaches 4 from the right. This

implies $\lim_{x\to 4^+} 1/(x-4)^5 = \infty$. Also, $\lim_{x\to 4^+} 10 - x^2 = -6$. This shows $\lim_{x\to 4^+} \dfrac{10-x^2}{(x-4)^5} = (-6)\infty = -\infty$.
As x approaches 4 from the left, $x-4$ approaches 0 and remains negative. This implies that $(x-4)^5$ approaches 0 and remains negative as x approaches 4 from the left. This implies $\lim_{x\to 4^-} 1/(x-4)^5 = -\infty$. Also, $\lim_{x\to 4^-} 10 - x^2 = -6$. This shows $\lim_{x\to 4^-} \dfrac{10-x^2}{(x-4)^5} = (-6)(-\infty) = \infty$.

(9) A continuous function f is defined by

$$f(x) = \begin{cases} x^2 + x & \text{if } x < a, \\ x^2 - 1 & \text{if } a \le x < b, \\ 9 - 4x^2 & \text{if } b \le x, \end{cases}$$

where a and b are real numbers such that $a < b$. Find a and b.

Continuity at a implies $\lim_{x\to a^-} f(x) = \lim_{x\to a} f(x) = f(a) = a^2 - 1$. The definition of $f(x)$ also implies $\lim_{x\to a^-} f(x) = \lim_{x\to a^-} x^2 + x = a^2 + a$. Combining these two assertions, we get $a^2 - 1 = a^2 + a$, which gives $a = -1$. Similarly, continuity at b implies $\lim_{x\to b^-} f(x) = \lim_{x\to b} f(x) = f(b) = 9 - 4b^2$. The definition of $f(x)$ also implies $\lim_{x\to b^-} f(x) = \lim_{x\to b^-} x^2 - 1 = b^2 - 1$. Combining the last two assertions, we get $9 - 4b^2 = b^2 - 1$, which gives $10 = 5b^2$. This implies $2 = b^2$ and $b = \pm\sqrt{2}$. We already showed $a = -1$. We have the assumption $a < b$. This gives $-1 = a < b$, which tells us that $b = -\sqrt{2}$ is impossible. But we have shown $b = \pm\sqrt{2}$. This implies that the $\pm$ is $+$, and we get $b = \sqrt{2}$.

(10) Evaluate $\quad \lim_{x\to 2} \dfrac{x^3-8}{x-2}\,, \quad \lim_{x\to 3} \dfrac{3-x}{\sqrt{3x+1}-\sqrt{10}}\,, \quad \lim_{x\to -\infty} \dfrac{x}{\sqrt{3x^2+7}}\,.$

Factorization gives us

$$\lim_{x\to 2} \frac{x^3-8}{x-2} = \lim_{x\to 2} \frac{(x-2)(x^2+2x+4)}{x-2} = \lim_{x\to 2}\ x^2 + 2x + 4 = 12.$$

Rationalization gives us

$$\begin{aligned} \lim_{x\to 3} \frac{3-x}{\sqrt{3x+1}-\sqrt{10}} &= \lim_{x\to 3} \frac{(3-x)(\sqrt{3x+1}+\sqrt{10})}{(\sqrt{3x+1}-\sqrt{10})(\sqrt{3x+1}+\sqrt{10})} \\ &= \lim_{x\to 3} \frac{(3-x)(\sqrt{3x+1}+\sqrt{10})}{(3x+1)-10} = \lim_{x\to 3} \frac{(3-x)(\sqrt{3x+1}+\sqrt{10})}{3(x-3)} \\ &= \lim_{x\to 3} -\frac{\sqrt{3x+1}+\sqrt{10}}{3} = -\frac{2\sqrt{10}}{3}. \end{aligned}$$

As x approaches $-\infty$, it becomes negative and we get $x = -|x| = -\sqrt{x^2}$. This gives

$$\begin{aligned}\lim_{x\to-\infty} \frac{x}{\sqrt{3x^2+7}} &= \lim_{x\to-\infty} \frac{-\sqrt{x^2}}{\sqrt{3x^2+7}} = \lim_{x\to-\infty} -\sqrt{\frac{x^2}{3x^2+7}} \\ &= -\sqrt{\lim_{x\to-\infty} \frac{x^2}{3x^2+7}} = -\sqrt{\frac{1}{3}} = -\frac{1}{\sqrt{3}} = -\frac{\sqrt{3}}{3}.\end{aligned}$$

(11) Prove $\lim\limits_{x\to 0}(x\cos^3(1/x)) = 0$ using the Squeeze Theorem.

We know $|\cos^3(1/x)| = |\cos(1/x)|^3 \le 1$ for all $x \neq 0$. This implies $|x\cos^3(1/x)| = |x|\cdot|\cos^3(1/x)| \le |x|$ for all $x \neq 0$. Now we know

$$|x\cos^3(1/x)| \le |x| \text{ for } x \neq 0.$$

This can be rewritten as

$$-|x| \le x\cos^3(1/x) \le |x| \text{ for } x \neq 0$$

because, in general, $|t| \le c$ is equivalent to $-c \le t \le c$. Now the Squeeze Theorem and the facts $\lim\limits_{x\to 0} -|x| = 0 = \lim\limits_{x\to 0}|x|$ lead to the conclusion $\lim\limits_{x\to 0}(x\cos^3(1/x)) = 0$.

(12) Evaluate $\displaystyle\lim_{x\to 0} \frac{\sin(7x)}{\tan(5x)}$ and $\displaystyle\lim_{x\to 0} \frac{x^2}{1-\cos x}$.

Using the substitution $u = 7x$ we obtain

$$\lim_{x\to 0}\frac{\sin(7x)}{x} = 7\lim_{x\to 0}\frac{\sin(7x)}{7x} = 7\lim_{u\to 0}\frac{\sin(u)}{u} = 7\cdot 1 = 7.$$

Using the substitution $v = 5x$ we obtain

$$\begin{aligned}\lim_{x\to 0}\frac{\tan(5x)}{x} &= 5\lim_{x\to 0}\frac{\tan(5x)}{5x} = 5\lim_{u\to 0}\frac{\tan(u)}{u} = 5\lim_{u\to 0}\frac{\frac{\sin(u)}{u}}{\cos u} = \frac{5\lim_{u\to 0}\frac{\sin(u)}{u}}{\lim_{u\to 0}\cos u} \\ &= \frac{5\cdot 1}{1} = 5.\end{aligned}$$

All this gives us

$$\lim_{x\to 0}\frac{\sin(7x)}{\tan(5x)} = \lim_{x\to 0}\frac{\frac{\sin(7x)}{x}}{\frac{\tan(5x)}{x}} = \frac{\lim_{x\to 0}\frac{\sin(7x)}{x}}{\lim_{x\to 0}\frac{\tan(5x)}{x}} = \frac{7}{5}.$$

From the fact $\displaystyle\lim_{x\to 0}\frac{\sin x}{x} = 1$ we get $\displaystyle\lim_{x\to 0}\frac{x}{\sin x} = \frac{1}{\lim_{x\to 0}\frac{\sin x}{x}} = \frac{1}{1} = 1$. This implies $\displaystyle\lim_{x\to 0}\frac{x^2}{\sin^2 x} = \lim_{x\to 0}\left(\frac{x}{\sin x}\right)^2 = 1^2 = 1$. Now we have

$$\begin{aligned}\lim_{x\to 0}\frac{x^2}{1-\cos x} &= \lim_{x\to 0}\frac{x^2(1+\cos x)}{(1-\cos x)(1+\cos x)} = \lim_{x\to 0}\frac{x^2(1+\cos x)}{1-\cos^2 x} = \lim_{x\to 0}\frac{x^2(1+\cos x)}{\sin^2 x} \\ &= \left(\lim_{x\to 0}\frac{x^2}{\sin^2 x}\right)\left(\lim_{x\to 0}(1+\cos x)\right) = 1\cdot(1+1) = 2.\end{aligned}$$

(13) Using the Intermediate Value Theorem, prove that the equation

$$x^5 + x^4 - 7x^3 + 2x^2 + 3x + \pi = 0$$

has a real solution x.

Let $f(x) = x^5 + x^4 - 7x^3 + 2x^2 + 3x + \pi$. Then $f(x)$ is a continuous function such that $f(-100) < 0 < f(100)$. The IVT tells us that there exists c in the interval $(-100, 100)$ such that $f(c) = 0$. Then $x = c$ is a real solution of the equation $x^5 + x^4 - 7x^3 + 2x^2 + 3x + \pi = 0$.

(14) Prove $\lim_{x \to 4} (3x - 5) = 7$ using an ε-δ argument.

Let $\varepsilon > 0$ be given. Choose $\delta = \varepsilon/3$. We know $\delta > 0$. If $0 < |x - 4| < \delta$ then $|(3x - 5) - 7| = |3x - 12| = |3(x - 4)| = |3| \cdot |x - 4| = 3 \cdot |x - 4| < 3\delta = 3(\varepsilon/3) = \varepsilon$.

(15) Prove $\lim_{x \to 0} \sqrt{|x|} = 0$ using an ε-δ argument.

Let $\varepsilon > 0$ be given. Choose $\delta = \varepsilon^2$. We know $\delta > 0$. If $0 < |x - 0| < \delta$ then $0 < |x| < \delta$, hence $|\sqrt{|x|} - 0| = \sqrt{|x|} < \sqrt{\delta} = \sqrt{\varepsilon^2} = \varepsilon$.

(16) Assume $f(x) = \sqrt{x}$. Prove $f'(x) = \dfrac{1}{2\sqrt{x}}$ using the definition

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

of the derivative.

We see

$$\begin{aligned}
\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} = \lim_{h \to 0} \frac{(\sqrt{x+h} - \sqrt{x})(\sqrt{x+h} + \sqrt{x})}{h(\sqrt{x+h} + \sqrt{x})} \\
&= \lim_{h \to 0} \frac{(x+h) - x}{h(\sqrt{x+h} + \sqrt{x})} = \lim_{h \to 0} \frac{h}{h(\sqrt{x+h} + \sqrt{x})} \\
&= \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{\sqrt{x} + \sqrt{x}} = \frac{1}{2\sqrt{x}}.
\end{aligned}$$

(17) Assume $f(x) = \dfrac{1}{\sqrt{x}}$. Prove $f'(x) = -\dfrac{1}{2x^{3/2}}$ using the definition

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

of the derivative.

We see

$$\begin{aligned}
\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} &= \lim_{h\to 0}\frac{1}{h}\cdot\left(\frac{1}{\sqrt{x+h}}-\frac{1}{\sqrt{x}}\right) = \lim_{h\to 0}\frac{1}{h}\cdot\frac{\sqrt{x}-\sqrt{x+h}}{\sqrt{x}\sqrt{x+h}} \\
&= \lim_{h\to 0}\frac{1}{h}\cdot\frac{(\sqrt{x}-\sqrt{x+h})(\sqrt{x}+\sqrt{x+h})}{\sqrt{x}\sqrt{x+h}(\sqrt{x}+\sqrt{x+h})} \\
&= \lim_{h\to 0}\frac{1}{h}\cdot\frac{x-(x+h)}{\sqrt{x}\sqrt{x+h}(\sqrt{x}+\sqrt{x+h})} \\
&= \lim_{h\to 0}\frac{1}{h}\cdot\frac{-h}{\sqrt{x}\sqrt{x+h}(\sqrt{x}+\sqrt{x+h})} \\
&= \lim_{h\to 0}\frac{-1}{\sqrt{x}\sqrt{x+h}(\sqrt{x}+\sqrt{x+h})} \\
&= -\frac{1}{\sqrt{x}\sqrt{x}(\sqrt{x}+\sqrt{x})} = -\frac{1}{2x^{3/2}}.
\end{aligned}$$

(18) Evaluate the following derivatives:

$$\frac{d}{dx}\sin^{-1}(e^x)\ ,\quad \frac{d}{dx}\tan^{-1}(\sqrt{x})\ ,\quad \frac{d}{dx}\sqrt{1+\sqrt{x}}\ ,$$

$$\frac{d}{dx}\left(x^2e^{\cos x}\right)\ ,\quad \frac{d}{dx}\left(\sec^{-1}(x)\right)^3\ ,\quad \frac{d}{dx}\frac{\tan(x^2+x^4)}{1+x^6}\ .$$

Using the Chain Rule and other Rules, we get

$$\frac{d}{dx}\sin^{-1}(e^x) = \frac{e^x}{\sqrt{1-(e^x)^2}} = \frac{e^x}{\sqrt{1-e^{2x}}},$$

$$\frac{d}{dx}\tan^{-1}(\sqrt{x}) = \frac{(1/2)x^{-1/2}}{1+(\sqrt{x})^2} = \frac{(1/2)x^{-1/2}}{1+x},$$

$$\frac{d}{dx}\sqrt{1+\sqrt{x}} = \frac{1}{2}(1+\sqrt{x})^{-1/2}\frac{1}{2\sqrt{x}},$$

$$\frac{d}{dx}\left(x^2e^{\cos x}\right) = 2xe^{\cos x} + x^2(-\sin x)e^{\cos x} = (2x - x^2\sin x)e^{\cos x},$$

$$\frac{d}{dx}\left(\sec^{-1}(x)\right)^2 = 2\sec^{-1}(x)\cdot\frac{1}{|x|\sqrt{x^2-1}},$$

$$\frac{d}{dx}\frac{\tan(x^2+x^4)}{1+x^6} = \frac{(\sec^2(x^2+x^4))(2x+4x^3)(1+x^6) - 6x^5\tan(x^2+x^4)}{(1+x^6)^2}.$$

(19) Find the second derivative of each of the following functions:

$$\sin^5 x\ , \quad \tan(\sqrt{x})\ , \quad e^{x^4+x}\ , \quad \sin^{-1}(x^2)\ , \quad \frac{1}{4+5x^2}\ .$$

If $f(x) = \sin^5 x$ then $f'(x) = 5\sin^4 x \cos x$ and

$$f''(x) = 5(4\sin^3 x\cos x)(\cos x) + 5\sin^4 x(-\sin x) = 20\sin^3 x\cos^2 x - 5\sin^5 x.$$

If $f(x) = \tan(\sqrt{x})$ then $f'(x) = \sec^2(\sqrt{x})(1/2)x^{-1/2} = (1/2)\sec^2(\sqrt{x})x^{-1/2}$ and

$$f''(x) = (1/2)2\sec(\sqrt{x})\sec(\sqrt{x})\tan(\sqrt{x})(1/2)x^{-1/2}x^{-1/2} + (1/2)\sec^2(\sqrt{x})(-1/2)x^{-3/2}.$$

If $f(x) = e^{x^4+x}$ then $f'(x) = (4x^3+1)e^{x^4+x}$ and

$$f''(x) = 12x^2e^{x^4+x} + (4x^3+1)(4x^3+1)e^{x^4+x} = (12x^2 + (4x^3+1)^2)e^{x^4+x}.$$

If $f(x) = \sin^{-1}(x^2)$ then $f'(x) = \dfrac{2x}{\sqrt{1-(x^2)^2}} = \dfrac{2x}{\sqrt{1-x^4}} = 2x(1-x^4)^{-1/2}$ and

$$f''(x) = 2(1-x^4)^{-1/2} + 2x(-1/2)(1-x^4)^{-3/2}(-4x^3)$$

If $f(x) = \dfrac{1}{4+5x^2}$ then $f(x) = (4+5x^2)^{-1}$, hence

$$f'(x) = (-1)(4+5x^2)^{-2}10x = -10x(4+5x^2)^{-2}$$

and

$$f''(x) = -10(4+5x^2)^{-2} - 10x(-2)(4+5x^2)^{-3}10x.$$

(20) Find the slope of the tangent line at the point $(1,2)$ on the graph of $x^2y^4 + xy = 18$. Implicit differentiation gives

$$2xy^4 + x^2(4y^3\frac{dy}{dx}) + y + x\frac{dy}{dx} = 0.$$

Substituting $x = 1$ and $y = 2$ gives

$$2\cdot 2^4 + 4\cdot 2^3\frac{dy}{dx} + 2 + \frac{dy}{dx} = 0.$$

Therefore, $\dfrac{dy}{dx} = -\dfrac{34}{33}$ when $x = 1$ and $y = 2$. The slope of the tangent line at the point $(1,2)$ on the graph of $x^2y^4 + xy = 18$ is $-\dfrac{34}{33}$