1. The proof that every planar graph is 5 colorable can be modified to the following (false) proof that every planar graph is 4 colorable.

As in the proof of the 5 color theorem, let $G$ be a counterexample with the fewest number of vertices and consider a planar embedding of $G$. We may assume that $G$ is maximal planar. Let $v$ be a vertex of degree at most 5 in $G$ and let $c$ be a 4-coloring of $G - v$. If the neighbors of $v$ are colored by $c$ with less than 3 colors then $c$ can be extended to a 4-coloring of $G$ so $c$ uses all four colors on $N(v)$. So $\deg(v) \geq 4$. If $v$ has degree 4 then the neighbors of $v$ induce a cycle $C$ with vertices $x_1, x_2, x_3, x_4$ in order and $x_i$ colored by color $i$. As in the proof of the 5CT there must be a path consisting of vertices colored 1 or 3 from $x_1$ to $x_3$ and a path consisting of vertices colored 2 or 4 from $x_2$ to $x_4$ and these paths lie in the outer face of $C$ which means they must have a vertex in common, a contradiction. Suppose $v$ has degree 5, so the neighbors of $v$ induce a cycle $C$ with vertices $x_1, x_2, x_3, x_4, x_5$ in order. Since all 4 colors appear on $C$, we may assume wlog that $x_i$ is colored $i$ for $1 \leq i \leq 4$ and $x_5$ is colored 2.

Then there is a color 1,3 path linking $x_1$ to $x_3$ outside of $C$ which means there is no color 2,4 path linking $x_2$ to $x_4$. Similarly there is a color 1,4 path linking $x_1$ to $x_4$ outside of $C$, but then there is no color 2,3 path linking $x_5$ to $x_3$. So recolor the 2,4 component of $x_2$ by interchanging colors 2 and 4, and recolor the 2,3 component containing $x_5$. Then we have a new coloring in which $x_2$ is color 4 and $x_5$ is color 3 and color 2 does not appear on the cycle $C$. So we can extend the coloring to a coloring of $G$ by coloring $v$ by 2.

Find a concrete example of a plane graph for which the above “recipe” for finding a 4-coloring fails.

2. Diestel, 5.6

3. Diestel, 5.16

4. Diestel, 5.17. (Prove your answer).

5. Diestel, 5.22.

6. In problem 5.22, we saw that for any $k$ there is a graph for which $\omega(G) = 2$ but $\chi(G) = k$ and hence the ratio $\chi(G)/\omega(G)$ can be arbitrarily large. In this problem we consider the question of how large the ratio $\chi(G)/\omega(G)$ can be as a function of the number of vertices of $G$. We prove: For all sufficiently large $n$ there is a graph on $n$ vertices with $\omega(G) \leq 2[\log_2 n]$ and $\chi(G) \geq \frac{n}{2[\log_2 n]}$. We will prove the existence of such a graph without actually describing a construction. Fix $n$ and observe first that it suffices to show that there is an $n$ vertex graph with no clique or independent set of size $2[\log_2 n]$

(a) Let $g(n)$ be the number of distinct graphs on the vertex set $V = \{1, \ldots, n\}$. Here we do NOT consider isomorphic graphs to be the same graph. Determine $g(n)$.

(b) Let $W$ be a subset of $V$ of size $k$ and let $f(n, W)$ be the number of graphs on $n$ vertices in which $W$ is a clique or an independent set. Determine $f(n, W)$.
(c) For \( k \leq n \), let \( h(n,k) \) be the number of graphs in which at least one subset of size \( k \) is a clique or an independent set. Show that if \( k \geq 2\lfloor \log n \rfloor \) then \( h(n,k) < g(n) \).

(d) Prove the theorem.

7. A simplicial vertex of a graph \( G \) is a vertex \( v \) with the property that its neighborhood induces a complete graph in \( G \). A (total) ordering \( v_1, \ldots, v_n \) of \( V(G) \) is a simplicial ordering if for all \( i \), \( v_i \) is a simplicial vertex of the subgraph induced on \( v_1, \ldots, v_{i-1} \).

(a) Prove that a graph is chordal if and only if it has a simplicial ordering. (Hint: use Proposition 5.5.1).

(b) Suppose that \( v_1, \ldots, v_n \) is a simplicial ordering of \( G \). Show that (i) the greedy coloring algorithm with this ordering yields an optimal coloring of \( G \), (ii) the greedy coloring algorithm for the complementary graph \( \overline{G} \) with the simplicial ordering need not yield an optimal coloring of \( \overline{G} \), (iii) the greedy coloring algorithm with the reverse of the simplicial ordering yields an optimal coloring of \( G \), (iv) the greedy algorithm with the reverse of the simplicial ordering need not yield the optimal coloring of \( G \).

8. (a) Prove that for any graph \( G \), \( \chi(G)\chi(\overline{G}) \geq |V(G)| \).

(b) Prove that for any graph \( G \), \( \chi(G) + \chi(\overline{G}) \geq 2\sqrt{|V(G)|} \).

(c) Show that for each perfect square \( n \) there is a graph on \( n \) vertices for which the previous two bounds are tight.

(d) Prove that \( \chi(G) + \chi(\overline{G}) \leq |V(G)| + 1 \).