Problems on polynomials

1. Suppose \( P(X) \in \mathbb{C}[X] \) is such that for every \( x \in \mathbb{R}, P(x) \in \mathbb{R} \). Show that all the coefficients of \( P(X) \) are real numbers.

2. Let \( P(X) = X^r + a_1X^{r-1} + \ldots + a_{r-1}X + a_r \) be a polynomial with complex coefficients such that \( P(X) \) divides \( X^n - 1 \). Show that \( |a_i| \leq \binom{n}{r} \).

3. Show that \( X^{100} - X - 5 \) cannot be written as the product of two polynomials of degree at least one with integer coefficients.

4. Find all complex numbers \( a, b \) such that the roots of \( x^2 + ax + b \) are \( \{a, b\} \).

5. Suppose \( P(X) \) is a polynomial such that \( P(1) = 0, dP/dX(1) = 0, \ldots, d^kP/dX^k(1) = 0 \). Then show that \( (X - 1)^k \) divides \( P(X) \).

6. Is there a polynomial with integer coefficients which has \( \sqrt{2} + \sqrt{3} \) as a root?

7. Find all polynomials \( P(X) \) such that \( P(P(X)) = X \).

8. Find all polynomials \( P(X) \) such that \( P(P(P(X))) = X \).

9. Show that for any real number \( a, b, c \),

\[
a^2 + b^2 + c^2 \geq ab + bc + ca.
\]

10. Show that

\[
\sum_{i=0}^{n-1} (i + 1) \binom{n}{i + 1} 2^i = n3^{n-1}.
\]

11. Let \( P(X) \) be a quadratic polynomial with real coefficients such that \( P(0), P(1) \) and \( P(2) \) are all integers. Then show that \( P(n) \) is an integer for all integers \( n \).

Also show that there are quadratic polynomials \( P(X) \) such that \( P(0), P(1) \) and \( P(3) \) are integers, but there are integers \( n \) for which \( P(n) \) is not an integer.

12. \( P(X) \) is a polynomial such that \( P(1) = 1 \) and \( P(-1) = 2 \). What is the remainder when you divide \( P(X) \) by \( X^2 - 1 \)?

13. Show that the polynomial

\[
P(X) = X^n + X^{n-1} + X^{n-2} + a_3X^{n-3} + a_4X^{n-4} + \ldots + a_n
\]

cannot have all its roots real.