Harmonic vectors and matrix tree theorems

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1 Introduction

In this paper we prove a new result in graph theory that was motivated by considerations in mathematical economics; more precisely by the problem of price formation in an exchange economy [3]. The aggregate demand/supply in the economy is described by an $n \times n$ matrix $A = (a_{ij})$ where $a_{ij}$ is the amount of commodity $j$ that is on offer for commodity $i$. In this context one defines a market-clearing price vector to be a vector $p$ with strictly positive components $p_i$, which satisfies the equation

$$\sum_j a_{ij} p_j = \sum_j a_{ji} p_i \text{ for all } i$$

(1)

The left side of (1) represents the total value of all commodities being offered for commodity $i$, while the right side represents the total value of commodity $i$ in the market. It was shown in [3] that if the matrix $A$ is irreducible, i.e. if it cannot be permuted to block upper-triangular form, then (1) admits a positive solution vector $p$, which is unique up to a positive multiple.

The primary purpose of the present paper is to describe an explicit combinatorial formula for $p$. The formula and its proof are completely elementary, but nonetheless the result seems to be new. This formula plays a crucial role in forthcoming joint work of the author [4], which seeks to address a fundamental question in mathematical economics: How do prices and money emerge in a barter economy? We show in [4] that among a reasonable class of exchange mechanisms, trade via a commodity money, even in the absence of transactions costs, minimizes complexity in a very precise sense.
It turns out however that equation (1) is closely related to well-studied problems in graph theory, in particular to the so-called matrix tree theorems. Therefore as an additional application of our formula, we give an elementary proof of the matrix tree theorem of W. Tutte [5], which was independently discovered by R. Bott and J. Mayberry [1] coincidentally also in an economic context. With a little additional effort, we also obtain a short new proof of S. Chaiken’s generalization of the matrix tree theorem [2].

2 Harmonic vectors

We first give a slight reformulation and reinterpretation of equation (1) in standard graph-theoretic language. Let $G$ be a simple directed graph (di-graph) on the vertices $1, 2, \ldots, n$, with weight $a_{ij}$ attached to the edge $ij$ from $i$ to $j$. The weighted adjacency matrix of $G$ is the $n \times n$ matrix $A = (a_{ij})$, where $a_{ij} = 0$ for missing edges. The degree matrix $D$ is the diagonal matrix with diagonal entries $(d_1, \ldots, d_n)$, where $d_i$ is the in-degree $\sum_j a_{ji}$ of the vertex $i$. The Laplacian of $G$ is the matrix $L = D - A$ and we say that a vector $x = (x_i)$ is harmonic if $x$ is a null vector of $L$, i.e. if it satisfies

$$Lx = 0.$$  \hfill (2)

It is easy to see that equation (1) is equivalent to equation (2), i.e. the market-clearing condition is the same as harmonicity of $p$.

To describe our construction of a harmonic vector, we introduce some terminology. A directed tree, also known as an arborescence, is a digraph with at most one incoming edge $ij$ at each vertex $j$, and whose underlying undirected graph is acyclic and connected (i.e. a tree). Following the edges backwards from any vertex we eventually arrive at the same vertex called the root. Dropping the connectivity requirement leads to the notion of a directed forest, which is simply a vertex-disjoint union of directed trees. We define a dangle to be a digraph $D$ that is an edge-disjoint union of a directed forest $F$ and a directed cycle $C$ linking the roots of $F$; note that $D$ determines $C$ and $F$ uniquely, the former as its unique simple cycle.

In the context of the digraph $G$, we will use the term $i$-tree to mean a directed spanning tree of $G$ with root $i$, and $i$-dangle to mean a spanning dangle whose cycle contains $i$. We define the weight $wt(\Gamma)$ of a subgraph $\Gamma$ of $G$ to be the product of weights of all the edges of $\Gamma$, and we define the weight vector of $G$ to be $w = (w_i)$ where $w_i$ is the weighted sum of all $i$-trees.
Theorem 1  The weight vector of a digraph is harmonic.

Proof. If $\Gamma$ is an $i$-dangle in $G$ with cycle $C$, and $ij$ and $ki$ are the unique outgoing and incoming edges at $i$ in $C$, then deleting one of these edges from $\Gamma$ gives rise to an $j$-tree and an $i$-tree, respectively. The dangle can be recovered uniquely from each of the two trees by reconnecting the respective edges; thus, writing $T_i$ for the set of $i$-trees, we obtain bijections from the set of $i$-dangles to each of the following sets

$$\{(ij, t) : t \in T_j\}, \quad \{(ki, t) : t \in T_i\}.$$ 

where $ij$ and $ki$ range over all outgoing and incoming edges at $i$ in $G$.

Thus if $v_i$ is the weighted sum of all $i$-dangles, we get

$$\sum_j a_{ij} w_j = v_i = \sum_k a_{ki} w_i.$$ 

Rewriting this we get $Aw = Dw$, and hence $(D - A)w = 0$, as desired. 

3 The matrix tree theorem

In this section we use Theorem 1 to derive the matrix tree theorem due to [5] (see also [1]). This is the following formula for the cofactors of the Laplacian $L$, which generalizes a classical formula of Kirchoff for the number of spanning trees in an undirected graph.

Theorem 2  The $ij$-th cofactor of the Laplacian $L$ is given by

$$c_{ij}(L) = \sum_{t \in T_j} \text{wt}(t) \text{ for all } i, j.$$ 

We will prove this in a moment after some discussion on cofactors.

3.1 Interlude on cofactors

We recall that $ij$-th cofactor of an $n \times n$ matrix $X$ is

$$c_{ij}(X) = (-1)^{i+j} \det X_{ij},$$

where $X_{ij}$ is the matrix obtained from $X$ by deleting row $i$ and column $j$. The adjoint of $X$ is the $n \times n$ matrix $\text{adj}(X)$ whose $ij$-th entry is $c_{ji}(X)$.
Lemma 3 If \( \det X = 0 \) then the columns of \( \text{adj}(X) \) are null vectors of \( X \); moreover these are the same null vector if the columns of \( X \) sum to 0.

Proof. By standard linear algebra we have \( X \text{adj}(X) = \det(X) I_n \). If \( \det X = 0 \) then \( X \text{adj}(X) \) is the zero matrix, which implies the first part. For the second part we note that if \( X \) has zero column sums then necessarily \( \det X = 0 \). In view of the first part it suffices to show that \( c_{ij}(L) = c_{i+1,j}(L) \) for all \( i, j \); or equivalently that

\[
\det(L_{ij}) + \det(L_{i+1,j}) = 0.
\]

The left side above equals \( \det P \), where \( P \) is obtained from \( L \) by deleting column \( j \) and replacing rows \( i \) and \( i+1 \) by the single row consisting of their sum. But \( P \) too has zero column sums, and so \( \det P = 0 \).

3.2 Proof of the matrix tree theorem

Proof. It suffices to prove Theorem 2 for the complete simple digraph \( G_n \) on \( n \) vertices, with edge weights \( \{a_{ij} \mid i \neq j\} \) regarded as variables, and we work over the field of rational functions \( \mathbb{C}(a_{ij}) \). The Laplacian \( L \) has zero column sums by construction, and so by the previous lemma, \( c_j := c_{ij}(L) \) is independent of \( i \) and the vector \( c = (c_1, \ldots, c_n)^t \) is a null vector for \( L \). To complete the proof it suffices to show that the null vectors \( c \) and \( w \) are equal. Now the null space of \( L \) is 1-dimensional since \( c_{ij}(L) \neq 0 \), and hence

\[
c_iw_j = c_jw_i \quad \text{for all } i, j. \tag{3}
\]

Note that \( c_j \) and \( w_j \) belong to the polynomial ring \( \mathbb{C}[a_{ij}] \). We claim that the polynomials \( c_j \) are distinct and irreducible. Consider first \( c_n = \det(B) \) where \( B = L_{nn} \) has entries

\[
b_{ij} = \begin{cases} -a_{ij} & \text{if } i \neq j, \\ a_{nj} + \sum_{k=1}^{n-1} a_{kj} & \text{if } i = j \end{cases}; \quad \text{for } 1 \leq i, j \leq n - 1.
\]

This is an invertible \( \mathbb{C} \)-linear map relating \( \{b_{ij}\} \) to the \((n - 1)^2\) variables

\[
\{a_{ij} \mid 1 \leq i \leq n, 1 \leq j \leq n - 1, i \neq j\},
\]

which occur in \( c_n \). Thus the irreducibility of \( c_n \) follows from the irreducibility of the determinant as a polynomial in the matrix entries [1, P. 176]. The argument for the other \( c_i \) is similar, and their distinctness is obvious.
Since $c_i$ and $c_j$ are distinct and irreducible, we conclude from (3) that $c_i$ divides $w_i$. Since $c_i$ and $w_i$ both have total degree $n - 1$, we conclude that $w_i = \alpha c_i$ for some $\alpha \in \mathbb{C}$. To prove that $\alpha = 1$, it suffices to note that the monomial $m_i = \prod_{j \neq i} a_{ij}$ occurs in both $c_i$ and $w_i$ with coefficient 1. ■

4 The all minors theorem

The all minors theorem [2] is a formula for $\det L_{IJ}$, where $L_{IJ}$ is the submatrix of $L$ obtained by deleting rows $I$ and columns $J$. It turns out this follows from Theorem 2 by a specialization of variables. We will state and prove this below after a brief discussion on signs of permutations and bijections.

4.1 Interlude on signs

Let $I, J$ be equal-sized subsets of $\{1, \ldots, n\}$ and let $\Sigma_I, \Sigma_J$ denote the sums of their elements. If $\beta : J \to I$ is a bijection, we write $\text{inv} (\beta)$ for the number of inversions in $\beta$, i.e. pairs $j < j'$ in $J$ such that $\beta (j) > \beta (j')$ and we define

$$
\varepsilon (\beta) = (-1)^{\text{inv} (\beta) + \Sigma_I + \Sigma_J}.
$$

Note that if $J = I$ then $\varepsilon (\sigma) = (-1)^{\text{inv} (\sigma)}$ is the sign of $\sigma$ as a permutation.

Lemma 4 If $\beta : J \to I$, $\alpha : I \to H$ are bijections then $\varepsilon (\alpha \beta) = \varepsilon (\alpha) \varepsilon (\beta)$.

Proof. This follows by combining the following mod 2 congruences

$$
\Sigma_H + \Sigma_I + \Sigma_I + \Sigma_J \equiv \Sigma_H + \Sigma_J, \quad \text{inv} (\alpha \beta) \equiv \text{inv} (\alpha) + \text{inv} (\beta),
$$

the first of which is obvious. To establish the second congruence we replace $\alpha, \beta$ by the permutations $\lambda \alpha, \beta \mu$ of $I$, where $\lambda : H \to I, \mu : I \to J$ are the unique order-preserving bijections; this does not affect $\text{inv} (\alpha)$ etc., and reduces the second congruence to a standard fact about permutations. ■

The meaning of $\varepsilon (\beta)$ is clarified by the following result. For a bijection $\beta : J \to I$ and any $n \times n$ matrix $X$, let $X_\beta$ be the matrix obtained from $X$ by replacing, for each $j \in J$, the $j$th column of $X$ by the unit vector $e_{\beta(j)}$.

Lemma 5 We have $\det X_\beta = \varepsilon (\beta) \det X_{IJ}$. 

5
Proof. If $\sigma$ is a permutation of $I$ then by the previous lemma, and standard properties of the determinant, we have

$$\varepsilon(\sigma\beta) = \varepsilon(\sigma)\varepsilon(\beta), \quad \det(X_{\sigma\beta}) = \varepsilon(\sigma)\det(X_{\beta})$$

Thus replacing $\beta$ by a suitable $\sigma\beta$, we may assume $\text{inv}(\beta) = 0$ and write

$I = \{i_1 < \cdots < i_p\}, J = \{j_1 < \cdots < j_p\}$ with $\beta(j_k) = i_k$ for all $k$.

The lemma now follows from the identity

$$\det(X_{\beta}) = (-1)^{i_1+j_1} \cdots (-1)^{i_p+j_p} \det(X_{IJ}) = (-1)^{\Sigma I + \Sigma J} \det(X_{IJ})$$

obtained by iteratively expanding $\det(X_{\beta})$ along columns $j_p, \ldots, j_1$. ■

### 4.2 Directed forests

Let $\mathcal{F}(J)$ be the set of all directed spanning forests $f$ of $G$ with root set $J$. Let $\mathcal{F} \subset \mathcal{F}(J)$ be the subset consisting of those forests $f$ such that each tree of $f$ contains a unique vertex of $I$. Note that the trees of $f \in \mathcal{F}$ give a bijection $\beta_f : J \to I$. The all minors theorem is the following formula [2].

**Theorem 6** We have $\det(L_{IJ}) = \sum_{f \in \mathcal{F}} \varepsilon(\beta_f) \text{wt}(f)$.

We fix a bijection $\beta : J \to I$ and define $\sigma_f = \beta^{-1}\beta_f : J \to J$. In view of Lemmas 4 and 5, it suffices to prove the following reformulation of the previous theorem.

**Theorem 7** We have $\det L_\beta = \sum_{f \in \mathcal{F}} \varepsilon(\sigma_f) \text{wt}(f)$.

**Proof.** As usual it is enough to treat the complete digraph $G_n$ with arbitrary edge weights $a_{ij}$. We fix an index $j_0 \in J$ and put $i_0 = \beta(j_0)$, $J_0 = J \setminus \{j_0\}$. We now consider a particular specialization $\bar{a}_{ij}$ of $a_{ij}$, and the entries $\bar{l}_{ij}$ of the specialized Laplacian $\bar{L}$. For $j \notin J_0$ we set $\bar{a}_{ij} = a_{ij}$ and hence $\bar{l}_{ij} = a_{ij}$; while for $j \in J_0$ we set

$$\bar{a}_{ij} = \begin{cases} 1 & \text{if } i = i_0 \\ -1 & \text{if } i = \beta(j) \\ 0 & \text{otherwise} \end{cases} \implies \bar{l}_{ij} = \begin{cases} -1 & \text{if } i = i_0 \\ 1 & \text{if } i = \beta(j) \\ 0 & \text{otherwise} \end{cases}$$

(4)
Note that $\bar{L}$ and $L_\beta$ have the same entries outside of row $i_0$ and column $j_0$; hence we get $\det L_\beta = c_{i_0j_0} (L_\beta) = c_{i_0j_0} (\bar{L})$ and it remains to show that

$$c_{i_0j_0} (\bar{L}) = \sum_{f \in \mathcal{F}} \varepsilon (\sigma_f) \operatorname{wt} (f).$$

(5)

Specializing Theorem 2 we get

$$c_{i_0j_0} (\bar{L}) = \sum_{f \in \mathcal{F}(J)} \psi (f) \operatorname{wt} (f), \quad \psi (f) := \sum_{t \in \mathcal{A}_f} (-1)^{p(t)},$$

where $\mathcal{A}_f$ is the set of $j_0$-trees $t$ such that for each $j \in J_0$ the unique edge $ij$ in $t$ satisfies $i = i_0$ or $i = \beta (j)$, and for which deleting all such edges from $t$ yields the forest $f$; and $p (t)$ is the number of edges in $t$ of type $i_0j, j \in J_0$. Therefore to prove (5) it suffices to show

$$\psi (f) = \begin{cases} 0 & \text{if } f \notin \mathcal{F} \\ \varepsilon (\sigma_f) & \text{if } f \in \mathcal{F} \end{cases}.$$

First suppose $f \notin \mathcal{F}$. In this case if $t \in \mathcal{A}_f$ then there is some $j \in J_0$ such that the $j$-subtree contains no $I$ vertex. Choose the largest such $j$ and change the edge $ij$, from $i = i_0$ to $i = \beta (j)$ or vice versa. This is a sign-reversing involution on $\mathcal{A}_f$ and hence we get $\psi (f) = 0$.

Now let $f \in \mathcal{F}$, and for each subset $S \subset J_0$ consider the graph obtained from $f$ by adding the edges $i_0j$ for $j \in S$, and $\beta (j)j$ for $j \in J_0 \setminus S$. This graph is a tree in $\mathcal{A}_f$ iff $S$ meets every cycle $c$ of the permutation $\sigma_f$ of $J$, and is disconnected otherwise. Thus a tree $t \in \mathcal{A}_f$ is prescribed uniquely by choosing, for each cycle $c$ of $\sigma_f$, a nonempty subset $S_c$ of its vertex set $J_c$. By definition we have $(-1)^{p(t)} = \prod_{c} (-1)^{|J_c|-|S_c|}$, and so $\psi (f)$ factors as

$$\psi (f) = \prod_{c} \psi (c), \quad \psi (c) := \sum_{J_c \supseteq S_c \neq \emptyset} (-1)^{|J_c|-|S_c|}.$$

Now we get $\psi (c) = (-1)^{|J_c|-1}$ using the elementary identity

$$\sum_{k=1}^{m} \binom{m}{k} (-1)^{m-k} = (1 - 1)^m - (-1)^m = (-1)^{m-1}.$$

Thus $\psi (f)$ agrees with the standard formula $\prod_{c} (-1)^{|J_c|-1}$ for $\varepsilon (\sigma_f)$. ■
References


