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THE CAPELLI IDENTITY FOR GRASSMANN MANIFOLDS

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Abstract. The column space of a real $n \times k$ matrix x of rank k is a k-plane. Thus we get a map from the space X of such matrices to the Grassmannian $\mathbb{G}$ of k-planes in $\mathbb{R}^n$, and hence a GL_n-equivariant isomorphism

$$C^{\infty}(\mathbb{G}) \approx C^{\infty}(X)^{GL_k}.$$

We consider the $O_n \times GL_k$-invariant differential operator C on X given by

$$C = \det\left(x^t x\right)\det\left(\partial^t \partial\right), \text{ where } x = (x_{ij}), \ \partial = \left(\frac{\partial}{\partial x_{ij}}\right).$$

By the above isomorphism C defines an O_n-invariant operator on $\mathbb{G}$.

Since $\mathbb{G}$ is a symmetric space for O_n, the irreducible O_n-submodules of $C^{\infty}(\mathbb{G})$ have multiplicity 1; thus O_n-invariant operators act by scalars on these submodules. Our main result determines these scalars for a general class of such operators including C. This answers a question raised by Howe and Lee [9] and also gives new Capelli-type identities for the orthogonal Lie algebra.

1. Introduction

1.1. The classical Capelli identity. Let $\{z_{ij} : 1 \le i, j \le n\}$ be n^2 variables and let $\partial_{ij} = \frac{\partial}{\partial z_{ij}}$ be the corresponding partial derivatives. Consider the $n \times n$ matrices $z = (z_{ij})$ and $\partial = (\partial_{ij})$. The classical Capelli identity [3] is the following identity of differential operators

$$\det(z)\det(\partial) = \det\left(z^t \partial + (n-i)\,\delta_{ij}\right). \tag{1.1}$$

Here δ_{ij} is the Kronecker δ function, and the (noncommuting) determinant on right is defined as follows:

$$\det(a_{ij}) = \sum_{w \in S_n} sgn(w)\, a_{w(1),1} \cdots a_{w(n),n}.$$

This identity plays a key role in the work of Herman Weyl [20] and others on classical invariant theory. It has also appeared more recently in the work of Atiyah-Bott-Patodi [1] on the index theorem. We refer the reader to ([7], [2]) for a discussion of the background of this identity. In the early 90s, the identity was generalized considerably by B. Kostant and the author ([13], [14], [19]) and by Howe-Umeda [10] to the context of Jordan algebras and multiplicity free actions. Since then there has been considerable interest in generalizations of Capelli-type identities, we refer the reader to ([4], [11], [12], [15], [16], [17], [18]) and to the references therein.

The representation-theoretic meaning of (1.1) was explained in [7]. The polynomial algebra $\mathbb{C}[z_{ij}]$ is the affine coordinate ring of the space $M_{n,n}$ of $n \times n$ matrices.

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The group $GL_n \times GL_n$ acts on $M_{n,n}$ by left and right multiplication, and hence we get a representation π on $\mathbb{C}[z_{ij}]$. The operator $\det(z)\det(\partial)$ on the left of (1.1) is a $GL_n \times GL_n$ invariant differential operator on $\mathbb{C}[z_{ij}]$. It follows from (GL_n, GL_n) duality [7] that such an operator is necessarily of the form $\pi(\Omega)$ for some element Ω in the center of the enveloping algebra of $\mathfrak{gl}_n \oplus \mathfrak{gl}_n$. Note however that the element Ω is not unique, and the expression on the right of (1.1) merely identifies one possible choice of Ω. Indeed the entries of the matrix $(z^t\partial)_{ij} = \sum_p z_{pi}\partial_{pj}$ come from the *right* action of GL_n alone, and one can find other similar expressions in terms of the matrix entries of the left action $(z\partial^t)_{ij} = \sum_p z_{ip}\partial_{jp}$, or the diagonal action $(z\partial^t - z^t\partial)_{ij}$ etc.

There is however an associated eigenvalue problem that does admit a unique answer. Again by (GL_n, GL_n) duality, or in this case by the Peter-Weyl theorem, one knows that $\mathcal{P}_{n,n}$ decomposes as a direct sum of $GL_n \times GL_n$-modules $\mathcal{P}_{n,n} = \bigoplus_\lambda \left(V^\lambda \otimes V^\lambda\right)$ where λ ranges over partitions of length $\leq n$ and V^λ is the GL_n representation with highest weight λ. By Schur's Lemma $\det(z)\det(\partial)$ acts by a scalar $p(\lambda)$ on $V^\lambda \otimes V^\lambda$ and one may ask for an explicit formula for this scalar. This turns out to have a rather pretty answer:

$$p(\lambda) = \prod_{i=1}^{n} (\lambda_i + n - i). \tag{1.2}$$

As explained in [13], the eigenvalue formula (1.2) implies (1.1) via the Harish-Chandra homomorphism. Moreover this point of view leads to a much larger class of identities in the context of Jordan algebras ([13], [14], [19]).

1.2. **The Capelli identity for Grassmann manifolds.** In [9] Howe and Lee pose a similar eigenvalue problem in the context of Grassmannians. In order to describe this problem we need some notation. Fix integers k, l with $k < l$ and write $n = k + l$. Let $\mathbb{G} = \mathbb{G}_{n,k}$ denote the Grassmannian of k planes in $\mathbb{C}^n$, and let $\mathcal{R} = \bigoplus_{m=0}^{\infty} \mathcal{R}_m$ be its homogeneous coordinate ring. The ring $\mathcal{R}$ admits an explicit description via the Plücker imbedding that we now recall briefly, referring the reader to [5, Ch. 9] for details. Let $\mathcal{P} = \mathbb{C}[z_{11}, \ldots, z_{nk}]$ be the coordinate ring of the space $M_{n,k}$ of $n \times k$ matrices, regarded as a $GL_n \times GL_k$-module. Then $\mathcal{R}$ is the subalgebra of $\mathcal{P}$ generated by the determinants of $k \times k$ minors. Alternatively we have

$$\mathcal{R} = \mathcal{P}^{SL_k}, \mathcal{R}_m = \mathcal{P}^{(GL_k, \det^m)}$$

with the usual notation for invariants and equivariants for the right action of GL_k on $M_{n,k}$. This description also shows that each $\mathcal{R}_m$ is naturally a module under the left action of GL_n. By (e.g. [7]) $\mathcal{R}_m$ has a multiplicity-free O_n-decomposition

$$\mathcal{R}_m = \bigoplus\nolimits_\mu \mathcal{R}_m^\mu$$

and the occurring summands are indexed by partitions of length $\leq k$, whose parts are less than m and have the same parity as m. In other words, μ ranges over integer sequences $\mu = (\mu_1, \cdots, \mu_k)$ satisfying

$$m \geq \mu_1 \geq \cdots \geq \mu_k \geq 0,\ \mu_i \equiv m \pmod 2 \text{ for all } i.$$

Consider now the $n \times k$ matrices $z = (z_{ij})$ and $\partial = \left(\frac{\partial}{\partial z_{ij}}\right)$, and define

$$\gamma(z) = \det\left(z^t z\right), L = \det\left(\partial^t \partial\right).$$

The differential operator L and the multiplication operator γ are O_n-invariant and transform under GL_k by the characters $\det^{-2}$ and $\det^2$ respectively. Thus they define maps

$$\gamma : \mathcal{R}_m^\mu \to \mathcal{R}_{m+2}^\mu, L : \mathcal{R}_m^\mu \to \mathcal{R}_{m-2}^\mu. \tag{1.3}$$

It follows that for any two integers $d, d' \geq 0$, the operator

$$C_{d,d'} = \gamma^d \circ L^{d+d'} \circ \gamma^{d'} \tag{1.4}$$

is an O_n-invariant operator on $\mathcal{R}_m^\mu$. By Schur's lemma $C_{d,d'}$ acts by a scalar on $\mathcal{R}_m^\mu$, and Howe-Lee [9] ask for an explicit formula for these scalar eigenvalues. To be precise they consider only the operator $L\gamma = C_{0,1}$, but the general case is not much more difficult as we explain below. For $k = 1$ the problem reduces to the classical theory of harmonic polynomials, and Howe-Lee [9] solve the problem for $k = 2$ by an explicit calculation. Their calculation is elementary but fairly intricate and it is unclear how to extend it for $k \geq 3$.

In this paper we solve the eigenvalue problem for all k using completely different methods. Let us define the following k-tuple of rational numbers

$$\rho = (\rho_1, \ldots, \rho_k) \in \mathbb{Q}^k, \ \rho_i = \frac{n}{2} - i. \tag{1.5}$$

Also let s and $\tau = (\tau_1, \ldots, \tau_k)$ be $k+1$ indeterminates and define polynomials

$$q_{1,0}(s, \tau) = \prod_{i=1}^{k} \left(s^2 - \tau_i^2\right), \ q_{d,d'}(s, \tau) = \prod_{j=0}^{d+d'-1} q_{1,0}(s + 2d' - 2j, \tau). \tag{1.6}$$

Our main result is as follows:

Theorem 1.1. *$C_{d,d'}$ acts on $\mathcal{R}_m^\mu$ by the scalar $q_{d,d'}(m + \rho_1, \mu + \rho)$.*

Specializing to $k = 2, d = 0, d' = 1$, we see that $q_{0,1}(m + \rho_1, \mu + \rho)$ equals

$$(m - \mu_1 + 2)(m + \mu_1 + n)(m - \mu_2 + 3)(m + \mu_2 + n - 1),$$

which agrees with Theorem 6.1 of [9], using formulas on [9, P. 356] for $\lambda = \mu$.

We now briefly sketch the proof of Theorem 1.1. In the next section, we reduce the problem to the special case of the operator $C = C_{1,0} = \gamma L$, and prove a simple lemma about the kernel of C. We also explain how to imbed the problem into the more general real analytic setting of the symmetric space $Y = SO_n(\mathbb{R}) / [SO_k(\mathbb{R}) \times SO_l(\mathbb{R})]$. The operator C gives rise to a family of invariant operators C_s on Y. The structure of invariant differential operators on a symmetric space is given by the Harish-Chandra homomorphism. In sections 3 and 4 we recall this theory and specialize it to Y. In section 5 we prove the eigenvalues of C_s are given by a polynomial function with suitable symmetry properties. Finally in section 6 we show that this polynomiality result, combined with the knowledge of the kernel of C proves Theorem 2.1 up to an overall constant, which we then prove to be 1 by an auxiliary computation.

In the appendix, we combine the results of this paper with those of [11] and [15] to obtain identities of invariant differential operators on the symmetric space $SO(p+q)/SO(p) \times SO(q)$. The main theorem also leads to a new inversion formula for the Radon transform on Grassmannians, which we plan to discuss in a subsequent paper.

2. Reduction of the problem

We first explain how deduce Theorem 1.1 from a special case. Define

$$C = C_{1,0} = \gamma L.$$

Theorem 2.1. *C acts on $\mathcal{R}_m^\mu$ by the scalar $q_{1,0}(m+\rho_1, \mu+\rho)$.*

Proof of Theorem 1.1. We abbreviate $q(m) = q_{1,0}(m+\rho_1, \mu+\rho)$. If v in $\mathcal{R}_m^\mu$ then $\gamma^{d'} v \in \mathcal{R}_{m+2d'}^\mu$, and by Theorem 2.1 we get

$$L\left(\gamma^{d'} v\right) = q(m+2d')\,\gamma^{d'-1} v.$$

Iterating this we get

$$L^{d+d'} \gamma^{d'} v = \left[\prod_{i=0}^{d+d'-1} q(m+2d'-2i)\right] \gamma^{-d} v.$$

We multiply both sides by γ^d to get the result. □

We now sketch the proof of Theorem 2.1. The following simple result on the kernel of C plays a key role.

Lemma 2.2. *If $\mu_1 = m$ then C acts by 0 on $\mathcal{R}_m^\mu$.*

Proof. As noted above $L : \mathcal{R}_m^\mu \to \mathcal{R}_{m-2}^\mu$. If $\mu_1 = m$ then $\mathcal{R}_{m-2}^\mu = 0$, thus L acts by 0 on $\mathcal{R}_m^\mu$ and so does $C = \gamma L$. □

In the next few sections we establish some general results about the eigenvalues of operators such as C, which enable us to deduce Theorem 2.1 from the above lemma. The key result is Proposition 3 below, which says that these eigenvalues are given by a polynomial function in the parameters m, μ with certain symmetry properties. In order to prove this proposition, it is important to extend the eigenvalue problem to a larger real analytic setting that we now explain.

The (holomorphic) polynomials in $\mathcal{P}$ and $\mathcal{R}$ are determined by their restrictions to the space $M_{n,k}(\mathbb{R})$ of real $n \times k$ matrices, and even to the open subset

$$X = M'_{n,k}(\mathbb{R})$$

of rank k matrices. We will write x_{ij} for the (real) coordinate functions on $M_{n,k}(\mathbb{R})$ and X, and regard $\mathcal{P}$ and $\mathcal{R}$ as spaces of polynomials in x_{ij}. Similarly we write $\partial_{ij} = \frac{\partial}{\partial x_{ij}}$ consider the restrictions of the various differential operators

$$x = (x_{ij}), \partial = (\partial_{ij}), \gamma(x) = \det\left(x^t x\right), L = \det\left(\partial^t \partial\right), C = \gamma L.$$

Let $H = GL_k^+(\mathbb{R}) = \{g \in GL_k(\mathbb{R}) \mid \det g > 0\}$ and define

$$Y = X/H.$$

The left action of the group $SO_n(\mathbb{R})$ on Y is transitive and allows one to identify Y with the compact symmetric space $SO_n(\mathbb{R}) / [SO_k(\mathbb{R}) \times SO_l(\mathbb{R})]$. The general structure of invariant differential operators on such a space is well understood, and we briefly recall the main ideas below.

3. Differential operators on symmetric spaces

In this section we briefly recall some well-known results on compact symmetric spaces, referring the reader to [6] for details and proofs.

Let K be a connected compact group with involution σ, let K^σ denote the σ-fixed subgroup and let $(K^\sigma)_0$ denote its identity component. A subgroup M of K is said to be a symmetric subgroup if $(K^\sigma)_0 \subset M \subset K^\sigma$, and in this case we say that K/M is a symmetric space. In the case of interest to us $M = (K^\sigma)_0$ so we make this simplifying assumption in the subsequent discussion. Let $\mathfrak{k}, \mathfrak{m}$ denote the complexified Lie algebras of K and M. The involution σ on K defines an involution, still denoted σ, on $\mathfrak{k}$, and we obtain the corresponding Cartan decomposition

$$\mathfrak{k} = \mathfrak{m} + \mathfrak{s}$$

into ± 1 eigenspaces of σ. We fix a maximal abelian subspace $\mathfrak{a} \subset \mathfrak{s}$ and let $\mathfrak{t}$ denote the centralizer of $\mathfrak{a}$ in $\mathfrak{m}$. Then $\mathfrak{h} = \mathfrak{a} + \mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{k}$. We let $\Sigma \subset \mathfrak{h}^*$ denote the root system of $\mathfrak{h}$ in $\mathfrak{k}$, with Weyl group $W = W(\Sigma)$, and fix a positive subsystem Σ^+. The irreducible representations of K are classified by their highest weight $\lambda \in \Lambda$, where $\Lambda \subset \mathfrak{h}^*$ denotes the lattice cone of dominant integral elements.

The representations of K with an M-fixed vector are those for which the highest weight μ belongs to the subcone $\Lambda_0 \subset \Lambda$ consisting of *even* integral elements that vanish on $\mathfrak{t}$. Thus if $\mathcal{H}$ denotes the space of K-finite functions on K/M then we have a multiplicity free decomposition into irreducible K-modules

$$\mathcal{H} = \bigoplus_{\mu \in \Lambda_0} \mathcal{H}^\mu .$$

It is convenient to work with the restricted root system $\Sigma_0 = \Sigma|_{\mathfrak{a}} \setminus \{0\} \subset \mathfrak{a}^*$, the restricted Weyl group W_0 and the positive subsystem $\Sigma_0^+ = \Sigma^+|_{\mathfrak{a}} \cap \Sigma_0$. We regard Λ_0 as a lattice contained in $\mathfrak{a}^*$ and we also define $\rho \in \mathfrak{a}^*$ by

$$\rho = \frac{1}{2} \sum_{\alpha \in \Sigma_0^+} m_\alpha \alpha,$$

where m_α denotes the multiplicity of the restricted root α.

Let $\mathbf{D}(K/M)$ denote the algebra of K-invariant differential operators on K/M. Then one has an isomorphism, the Harish-Chandra isomorphism

$$D_0 \mapsto \Gamma_{D_0} : \mathbf{D}(K/M) \to \mathcal{I}(\mathfrak{a}),$$

where $\mathcal{I}(\mathfrak{a}) \approx \mathcal{S}(\mathfrak{a})^{W_0}$ denotes the algebra of W_0-invariant elements in the symmetric algebra $\mathcal{S}(\mathfrak{a})$. We regard $S(\mathfrak{a})$ and $\mathcal{I}(\mathfrak{a})$ as spaces of the polynomial functions on $\mathfrak{a}^*$, and we write $\mathcal{I}_d(\mathfrak{a}) \subset \mathcal{I}(\mathfrak{a})$ for the space of invariant polynomials of degree $\leq d$. The key property of the Harish-Chandra homomorphism is as follows:

Proposition 1. *If $D_0 \in \mathbf{D}(K/M)$ has order $\leq d$, then $\Gamma_{D_0} \in \mathcal{I}_d(\mathfrak{a})$. Moreover D_0 acts on $\mathcal{H}^\mu$ by the scalar $\Gamma_{D_0}(\mu + \rho)$.*

4. Differential operators on Grassmannians

We specialize the results of the previous section to the case

$$K = SO_n(\mathbb{R}), M = SO_k(\mathbb{R}) \times SO_l(\mathbb{R}), Y = K/M,$$

assuming as before that $n = k + l$ and $k < l$. In this case the restricted root system is of type B_k and we choose the usual positive subsystem

$$\Sigma_0 = \{\pm e_i \pm e_j, \pm e_j\}, \Sigma_0^+ = \{e_i \pm e_j \mid i < j\} \cup \{e_i\},$$

where $e_1, \ldots, e_k$ are the unit vectors in $\mathfrak{a}^* \approx \mathbb{C}^k$. The root multiplicities are 1 for $e_i \pm e_j$, and $(l-k)$ for e_j. Thus we get

$$\rho = \frac{1}{2} \sum_i \left[2(k-i) + (l-k)\right] e_i = \sum_i \left[\frac{n}{2} - i\right] e_i,$$

which we note agrees with formula (1.5). The lattice cone Λ_0 consists of integer k-tuples $\mu = (\mu_1, \ldots, \mu_k)$ satisfying

$$\mu_1 \geq \cdots \geq \mu_k \geq 0, \mu_i \equiv \mu_j \pmod 2 \text{ for all } i, j.$$

The Weyl group W_0 acts on $\mathbb{C}^k$ by sign changes and permutations of the coordinates. Thus

$$W_0 \approx S_k \ltimes (\mathbb{Z}/2)^k$$

and $\mathcal{I} = \mathcal{I}(\mathfrak{a})$ is the ring of polynomials in k variables, invariant under permutations and sign changes. Now Proposition 1 specializes as follows:

Proposition 2. *Let $D_0 \in \mathbf{D}(Y)$ be an $SO_n(\mathbb{R})$-invariant differential operator of order $\leq d$, then D_0 acts on the space $\mathcal{H}^\mu$ by the scalar $\Gamma_{D_0}(\mu + \rho)$, where $\Gamma_{D_0}(\tau)$ is a polynomial of degree $\leq d$ in k variables $\tau = (\tau_1, \ldots, \tau_k)$ which is invariant under permutations and sign changes.*

5. Twisted differential operators

We continue with the notation $K = SO_n(\mathbb{R}), M = SO_k(\mathbb{R}) \times SO_l(\mathbb{R})$ and $Y = K/M$ as in the previous section, and also recall from section 2 that

$$X = M'_{n,k}(\mathbb{R}), H = GL_k^+(\mathbb{R}), Y = X/H.$$

Let D be a $K \times H$ invariant differential operator of order d on X. Then D descends to a K-invariant differential operator D_0 on $Y = K/M$, whose Harish-Chandra image Γ_{D_0} belongs to the space $\mathcal{I}_d = \{p \in \mathcal{I} \mid \deg(p) \leq d\}$. Now the function $\gamma = \det(x^t x)$ is strictly positive on X, hence for any real (or complex) number s, we can consider the twisted differential operator $\gamma^{-s} \circ D \circ \gamma^s$, which descends to a K-invariant differential operator D_s on Y.

Theorem 5.1. *If D is a $K \times H$ invariant differential operator of order $\leq d$ on X, then the map $p_D : \mathbb{R} \to \mathcal{I}_d$ defined by $p_D(s) = \Gamma_{D_s}$ is a polynomial map in s of degree $\leq d$. More precisely, there exist elements $c_j = c_{j,D} \in \mathcal{I}_j$ such that $p_D(s) = \sum_{j=0}^d c_j s^{n-j}$.*

For the proof we need the following simple result:

Lemma 5.2. *Let V be a finite dimensional vector space and let $p : \mathbb{R} \to V$ be a smooth function such that for all t, $p(s+t) - p(s)$ is a polynomial of degree $\leq d-1$ in s. Then p is a polynomial function of degree $\leq d$.*

Proof. Taking the d^{th} order partial derivative of $p(s+t) - p(s)$ with respect to s, we deduce that for all s, t

$$p^{(d)}(s+t) - p^{(d)}(s) = 0.$$

Substituting $s = 0$, we conclude that $p^{(d)}(t) = p^{(d)}(0)$ is a constant for all t. It follows that $p^{(d+1)}(t) = 0$, whence p must be a polynomial of degree $\leq d$. $\square$

Proof of Theorem 5.1. We proceed by induction on d. The result is clear for $d=0$, since an operator of order 0 is multiplication by a function, which by virtue of $K\times H$-invariance reduces to a constant, and hence belongs to $\mathcal{I}_0$. We now assume that $d>0$ and that the result holds for all invariant operators of order $\leq d-1$. Fix $t\in\mathbb{R}$ and define

$$D'=\gamma^{-t}\circ D\circ\gamma^{t}-D=\gamma^{-t}\left[D,\gamma^{t}\right].$$

Then D' is invariant by the first expression and is of order $\leq d-1$ by the last expression. Therefore by induction $p_{D'}(s)$ is a polynomial function of degree $\leq d-1$ taking values in $\mathcal{I}_{d-1}$. However $D'_s=D_{s+t}-D_s$ and so

$$p_{D'}(s)=\Gamma_{D_{s+t}}-\Gamma_{D_s}=p_D(s+t)-p_D(s).$$

Thus we conclude that for all t, $p_D(s+t)-p_D(s)$ is a polynomial of degree $\leq d-1$ in s. From the definition, it is easy to see that $p_D(s)$ is a smooth function of s. It follows from the previous lemma that p_D is a polynomial of degree $\leq d$ in s.

Thus we have an expression $p_D(s)=\sum_{j=0}^{d}c_js^{d-j}$ with $c_j\in\mathcal{I}_d$ and it remains to show that $c_j\in\mathcal{I}_j$ for all $j\leq d-1$. We proceed once again by induction on d. The case $d=0$ follows as in the previous paragraph. Let us consider the expansion of $h(s)=p_D(s+1)-p_D(s)$

$$h(s)=\sum_{j=0}^{d-1}c'_js^{d-1-j}=\sum_{j=0}^{d}c_j(s+1)^{d-j}-\sum_{j=0}^{d}c_js^{d-j}.$$

It follows that $c'_j=(d-j)\,c_j+$ a combination of c_k for $k<j$. Inverting this we deduce that for $j\leq d-1$, we have

$$c_j=\frac{1}{d-j}c'_j+\text{a combination of }c'_k\text{ for }k<j.$$

Now as explained in the previous paragraph, the polynomial h is of the form $p_{D''}$ for the invariant operator $D''=\gamma^{-1}\circ D\circ\gamma-D$ of degree $d-1$. Therefore by induction $c'_j\in\mathcal{I}_j$ for all $j\leq d-1$, and by the above expression for c_j we conclude $c_j\in\mathcal{I}_j$ as well. $\square$

Proposition 3. *If D is a $K\times H$ invariant differential operator of order $\leq d$ on X, then there is a polynomial $p_D(s,\tau)$ of total degree $\leq d$ such that D_s acts on $\mathcal{H}^\mu$ by the scalar $p_D(s,\mu+\rho)$. Moreover p_D is invariant under permutations $\tau_i\leftrightarrow\tau_j$ and sign changes $\tau_i\to-\tau_i$.*

Proof. This follows by combining Theorem 5.1 and Proposition 2. $\square$

6. Proof of Theorem 2.1

We apply the result of the previous section to the invariant differential operator $C=C_{1,0}=\det(x^tx)\det(\partial^t\partial)$, which has degree $2k$. By the previous proposition there exists a polynomial $p_C(s,\tau)$ of degree $\leq 2k$ such that C_s acts on $\mathcal{H}^\mu$ by the character $p_C(s,\mu+\rho)$.

Proposition 4. *C acts on $\mathcal{R}^\mu_m$ by the character $p_C\left(\frac{m}{2},\mu+\rho\right)$.*

Proof. Let $f\in\mathcal{R}^\mu_m$, then $\gamma^{-m/2}f$ is H-invariant, hence $\gamma^{-m/2}f\in\mathcal{H}^\mu$. Evidently $\gamma^{-s}C=C_s\gamma^{-s}$, thus by the previous corollary we get

$$\gamma^{-m/2}Cf=C_{m/2}\left[\gamma^{-m/2}f\right]=p_C\left(\frac{m}{2},\mu+\rho\right)\left[\gamma^{-m/2}f\right].$$

Multiplying both sides by $\gamma^{m/2}$ we get the desired result. $\square$

In view of the above result, let us define

$$q(s,\tau) = p_C\left(\frac{s-\rho_1}{2},\tau\right). \tag{6.1}$$

Then by the previous proposition C acts on $\mathcal{R}_m^\mu$ by the scalar $q(m+\rho_1,\mu+\rho)$, and Theorem 2.1 reduces to showing the following result.

Theorem 6.1. *We have $q = q_{1,0}$.*

We will prove this in two stages.

Proposition 5. *There is a constant a such that $q = aq_{1,0}$.*

Proof. Dividing q by $s-\tau_1$ we obtain an expression

$$q(s,\tau) = (s-\tau_1)\,h(s,\tau) + r(\tau). \tag{6.2}$$

Let $\mu \in \Lambda_0$ be arbitrary, then Lemma 2.2 implies that C acts on $\mathcal{R}_m^\mu$ by 0 for $m = \mu_1$. Thus we get

$$q(\mu_1+\rho_1,\mu+\rho) = 0.$$

Substituting $s = \mu_1+\rho_1$ and $\tau = \mu+\rho$ in formula (6.2) we deduce that for all $\mu \in \Lambda_0$

$$r(\mu+\rho) = 0.$$

Now it is easy to see by induction on k that the set $\Lambda_0+\rho$ is Zariski dense in $\mathbb{C}^k$. It follows that $r(\tau)$ is identically 0, and thus $(s-t_1)$ divides q. Since p_C and hence q are invariant under permutations $\tau_i \leftrightarrow \tau_j$ and sign changes $\tau_i \to -\tau_i$, it follows that $(s\pm\tau_i)$ divides q for all i. Thus $q_{1,0} = \prod_{i=1}^k\left(s^2-\tau_i^2\right)$ divides q, but since $\deg q \le 2k$ we must have $q(s,\tau) = aq_{1,0}$ for some constant a. $\square$

Now Theorem 2.1 reduces to proving $a = 1$. For the proof we need a simple result on polynomials. In order to formulate and prove this result is convenient to proceed in somewhat greater generality, thus we temporarily introduce the following notation:

$$x = (x_1,\ldots,x_n),\partial_i = \partial/\partial x_i,\partial = (\partial_1,\ldots,\partial_n).$$

Let $g(x)$ be a polynomial, let s be an additional variable and define $g^s(x) = [g(x)]^s$, considered as an analytic function on a suitable open set in (x,s). Let $f(x)$ be a homogenous polynomial of degree d and regard $f(\partial)$ as a constant coefficient differential operator of order d. We apply $f(\partial)$ to g^s and then multiply the result by g^{d-s} to obtain the expression

$$g^{d-s}\left(f(\partial)\,g^s\right).$$

It is easy to see that this expression is a polynomial in (x,s). We now have the following result where we use the symbol $\sim$ to denote equality modulo lower degree terms in s.

Lemma 6.2. *In the setting of the previous paragraph, we have*

$$g^{d-s}\left(f(\partial)\,g^s\right) \sim s^d f(\nabla g),$$

where $\nabla g = (\partial_1 g,\ldots,\partial_n g)$ denotes the gradient of g.

Proof. It suffices to prove the result when f is a monomial. Thus we need to show that

$$g^{d-s}\left[\partial_{i_1}\cdots\partial_{i_d}g^s\right]\sim s^d\left(\partial_{i_1}g\right)\cdots\left(\partial_{i_d}g\right),$$

which follows easily by the chain rule and induction on d. □

We now return to the general discussion.

Proposition 6. *The constant a in Proposition 5 is* 1*.*

Proof. Let $\mathbf{1}$ denote the constant function in $\mathcal{H}^0$. By the previous proposition we have

$$C_s\mathbf{1}=q\left(2s+\rho_1,\rho\right)\mathbf{1}=aq_{1,0}\left(2s+\rho_1,\rho\right)\mathbf{1}\sim\left(a4^k\right)s^{2k}\mathbf{1}. \tag{6.3}$$

On the other hand, recalling the definition of C_s, we get

$$\gamma^{2k-1}\left[C_s\mathbf{1}\right]=\gamma^{2k-s}\left[\det\left(\partial^t\partial\right)\left(\gamma^s\right)\right]\sim s^{2k}\det\left(y^ty\right), \tag{6.4}$$

after applying the pevious lemma and writing $y=\nabla\gamma$.

Now multiplying (6.3) by γ^{2k-1} and comparing with (6.4) we get

$$s^{2k}\det\left(y^ty\right)\sim s^{2k}\left(a4^k\right)\gamma^{2k-1}. \tag{6.5}$$

Let I_k denote the $k\times k$ identity matrix, and let J denote the $n\times k$ matrix $\left[\begin{smallmatrix}I_k\\0\end{smallmatrix}\right]$, then we have $J^tJ=I_k,\gamma\left(J\right)=1$. Moreover an easy calculation shows that

$$y\left(J\right)=\nabla\gamma\left(J\right)=2J.$$

Thus evaluating both sides of formula (6.5) at $x=J$ we get

$$\det\left(4I_k\right)s^{2k}\sim\left(a4^k\right)s^{2k}.$$

Since $\det\left(4I_k\right)=4^k$, this proves $a=1$ as desired. □

As noted above, this proves Theorem 6.1 and hence Theorem 2.1.

7. Appendix

We apply the results of this paper to obtain identities in the algebra $\mathbf{D}\left(Y\right)$ of K-invariant differential operators on the symmetric space $Y=K/M$, where

$$K=SO_n\left(\mathbb{R}\right),M=SO_k\left(\mathbb{R}\right)\times SO_{n-k}\left(\mathbb{R}\right).$$

As noted in section 5 above we may also write $Y=X/H$, where

$$X=M'_{n,k}\left(\mathbb{R}\right),H=GL_k^+\left(\mathbb{R}\right).$$

Thus if D is a $K\times H$ invariant differential operator on X, then for any real (or complex) number s, the twisted differential operator $\gamma^{-s}\circ D\circ\gamma^s$ descends to an operator D_s in $\mathbf{D}\left(Y\right)$. This applies in particular to the element $C_{d,d'}=\gamma^d\circ L^{d+d'}\circ\gamma^{d'}$ in (1.4), and we write

$$C_{d,d',s}=\gamma^{-s}\circ C_{d,d'}\circ\gamma^s=\gamma^{d-s}\circ L^{d+d'}\circ\gamma^{d'+s}.$$

On the other hand if z belongs to the center $\mathcal{Z}\left(\mathfrak{k}\right)$ of the enveloping algebra $\mathcal{U}\left(\mathfrak{k}\right)$ of $\mathfrak{k}=\mathfrak{so}\left(n\right)$, then z gives rise to an operator $\mathbf{D}\left(Y\right)$ as well. A number of authors have given explicit construction of elements in $\mathcal{Z}\left(\mathfrak{k}\right)$. In particular we refer the reader to [11, Sections 6,7] for a certain element $D^\dagger\left(\lambda\right)=D_F\left(\lambda\right)\in\mathcal{Z}\left(\mathfrak{k}\right)$ defined in terms of a noncommuting determinant, and shown in [11, P. 487] to coincide with another element constructed in terms of the Sklyanin determinant in [15].

Theorem 7.1. *The following identity holds in* $\mathbf{D}(Y)$

$$C_{d,d',s} = a_{d,d'}(s) \prod_{j=0}^{d+d'-1} D^{\dagger}\left(2\left(s+d'-j\right)+n/2-1\right),$$

where $a_{d,d'}(s)$ *is a certain explicit constant.*

Proof. Writing $C = C_{1,0} = \gamma \circ L$ and $C_s = \gamma^{1-s} \circ L \circ \gamma^s$ we have a factorization

$$C_{d,d',s} = C_{s-d+1} \ldots C_{s+d'} = \prod_{j=0}^{d+d'-1} C_{s+d'-j}.$$

Thus it suffices show that for some explicit constant $a(s)$ one has

$$C_s = a(s)\, D^{\dagger}\left(2s+n/2-1\right). \tag{7.1}$$

As before let $\mathcal{H}$ denote the space of K-finite functions on K/M, and let $\mathcal{H} = \bigoplus_{\mu\in\Lambda_0} \mathcal{H}^{\mu}$ denote its (multiplicity-free) decomposition into irreducible K-modules. It suffices to show that the two sides of (7.1) agree on each $\mathcal{H}^{\mu}$. Let $\rho_i = n/2 - i$ as before, then by (6.1) in section 6, C_s acts on $\mathcal{H}^{\mu}$ by the scalar

$$p_C(s, \mu+\rho) = q(2s+\rho_1, \mu+\rho) = \prod\nolimits_{i=1}^{k}\left[\left(2s+\rho_1\right)^2 - \left(\mu_i+\rho_i\right)^2\right]. \tag{7.2}$$

Now the action of $D^{\dagger}(\lambda)$ on $\mathcal{H}^{\mu}$ is given via the Harish-Chandra homomorphism

$$\bar{\gamma} : \mathcal{Z}(\mathfrak{k}) \to \mathcal{S}(\mathfrak{h}),$$

where $\mathfrak{h}$ is a maximal toral subalgebra in $\mathfrak{k}$, followed by evaluation at

$$\left(\mu_1+\rho_1, \ldots, \mu_k+\rho_k, \rho_{k+1}, \ldots, \rho_p\right),$$

where $p = \lfloor n/2 \rfloor$. By Theorem 7.2 of [11] we see that $D^{\dagger}(\lambda)$ acts on $\mathcal{H}^{\mu}$ by

$$\begin{array}{ll} \prod_{i=k+1}^{p}\left(\lambda^2-\rho_i^2\right)\prod_{i=1}^{k}\left[\lambda^2-\left(\mu_i+\rho_i\right)^2\right] & \text{if} \quad n \text{ is even}, \\ \left(\lambda-1/2\right)\prod_{i=k+1}^{p}\left(\lambda^2-\rho_i^2\right)\prod_{i=1}^{k}\left[\lambda^2-\left(\mu_i+\rho_i\right)^2\right] & \text{if} \quad n \text{ is odd}. \end{array}$$

Comparing this with (7.2) we obtain (7.1), which implies the result. □

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