1 Lecture 1 (1/18/2011)

Definition 1 *A* proposition *is a sentence that is either true (T) or false (F).*

Example 2 $\boxed{1+2=3}$ *and* $\boxed{1+2=4}$ *are propositions.* *Tutankhamen had a lisp* *is also a proposition even though we do not know its* truth value*. However* *Who was that?* *is not a proposition, and neither is* $\boxed{x=y}$*, unless x and y have been defined previously.*

Definition 3 *If P, Q are propositions then we define three new propositions as follows:*

1. *$P \wedge Q$ ("P and Q"), which is true if and only if P, Q are both true.*
2. *$P \vee Q$ ("P or Q"), which is false if and only if P, Q are both false.*
3. *$\sim P$ ("not P"), which is true if and only if P is false.*

Example 4 *Black is white* and $1+2=3$ *is false.* *Black is white* or $1+2=3$ *is true.* It is not the case that *black is white* *is true.*

"and", "or", "not" ($\wedge$, $\vee$, $\sim$) are called *connectives*; propositions formed using connectives are called *compound* propositions.

Definition 5 *A* (propositional) formula *is an expression constructed using*

1. *finitely many variables $P_1, P_2, \cdots, P_n$ that represent propositions*
2. *finitely many connectives $\wedge$, $\vee$, $\sim$*
3. *finitely many grouping symbols such as parentheses*

with the property that if we assign a truth value to each variable then the formula acquires a well-defined truth value using the rules of Definition 3.

Example 6 *$P \wedge Q$, $P \vee Q$, $\sim P$ are all formulas. $(P \wedge Q) \vee R$, $P \wedge (Q \vee R)$ are both formulas, but $P \wedge Q \vee R$ is not a valid formula. (Why not?)*

Definition 7 *The* truth table *of a formula $f(P_1, \cdots, P_n)$ is a table that has one column for each variable plus one column for f; it has one row for each possible truth value assignment of the variables along with the corresponding truth value of the formula. If $f_1, f_2, \cdots$ are several formulas we can construct a* simultaneous *truth table by including one column for each formula.*

Example 8 *The simultaneous truth table for $P \wedge Q$, $P \vee Q$, $\sim P$ is as follows*

P	Q	$P \wedge Q$	$P \vee Q$	$\sim P$
T	T	T	T	F
F	T	F	T	T
T	F	F	T	F
F	F	F	F	T

Note that even though $\sim P$ does not involve Q, we can include it in the above table by ignoring Q. We can also draw a separate smaller table for $\sim P$.

P	$\sim P$
T	F
F	T

Definition 9 *A formula $f(P_1, \cdots, P_n)$ is called a* tautology *if it is always true, i.e. it is true for every choice of truth values of the variables $P_1, \cdots, P_n$. A formula $f(P_1, \cdots, P_n)$ is called a* contradiction *if it is always false.*

Example 10 *$P \vee (\sim P)$ is a tautology, $P \wedge (\sim P)$ is a contradiction.*

P	$\sim P$	$P \vee (\sim P)$	$P \wedge (\sim P)$
T	F	T	F
F	T	T	F

Definition 11 *Two formulas $f(P_1, \cdots, P_n)$ and $g(P_1, \cdots, P_n)$ are said to be* equivalent *if they have the same truth table. A* denial *of a formula $f(P_1, \cdots, P_n)$ is a formula equivalent to $\sim f(P_1, \cdots, P_n)$.*

Example 12 *$(\sim P) \wedge (\sim Q)$ is a denial of $P \vee Q$*

P	Q	$\sim P$	$\sim Q$	$(\sim P) \wedge (\sim Q)$	$P \vee Q$	$\sim (P \vee Q)$
T	T	F	F	F	T	F
F	T	T	F	F	T	F
T	F	F	T	F	T	F
F	F	T	T	T	F	T

1.1 Exercises

1. Show using a truth table that $(\sim P) \vee (\sim Q)$ is a denial of $P \wedge Q$.

2. Show using a truth table that $(P \wedge Q) \vee R$, $P \wedge (Q \vee R)$ are not equivalent.

3. The *xor (exclusive or)*, denoted $\oplus$, is a connective defined so that $P \oplus Q$ is true if and only if exactly one of P, Q is true. Show using a truth table that $P \oplus Q$ is equivalent to $(P \vee Q) \wedge \sim (P \wedge Q)$.

4. Let $f(P, Q)$ and $g(P, R)$ be two tautologies; show that f, g are equivalent by considering a simultaneous truth table in P, Q, R. Are any two tautologies equivalent, *even if they involve different variables*?