# 1 Lecture 5 (2/3/2011)

**Key Terms**: *Theorem, Proof, Modus Ponens,*
**Key Symbols:** $\vdash$

## 1.1 Proofs

**Definition 1** *A* theorem *is a statement of the form "If $H_1, \cdots, H_n$ are true then $Q$ is true". Here the $H_i$ are propositions called* hypotheses *and $Q$ is a proposition called the* conclusion*.*

**Definition 2** *A proof of a theorem is a list of propositions $P_1$, $P_2$, $\cdots$, $P_k$, where the last proposition is $P_k = Q$, and each $P_i$ is one of the following:*

1. *A hypothesis $(H)$*
2. *A tautology $(T)$*
3. *Obtained from* two *preceding propositions by the* Modus Ponens *rule $(M)$*

The Modus Ponens rule simply says "*if the propositions $A$ and $A \Rightarrow B$ have previously occurred in the list then we are allowed to write down $B$*".

It is a good idea to think of $P_1$, $P_2$, $\cdots$, $P_k$ as the moves of a solitaire game (or a maze) in which objective is to reach the conclusion $Q$. Conditions 1, 2, 3 are the rules of the game that tell us whether a move is legal or not.

If $Q$ is provable from $H_1, \cdots, H_n$ then we write

$$H_1, \cdots, H_n \vdash Q.$$

We now look at some simple examples.

**Theorem 3 (E)** *If $P$, "$P$ is equivalent to $Q$" are true then $Q$ is true.*

**Proof.**

| $P$ | $P \Leftrightarrow Q$ | $(P \Leftrightarrow Q) \Rightarrow (P \Rightarrow Q)$ | $P \Rightarrow Q$ | $Q$ |
|---|---|---|---|---|
| H | H | T | M | M |

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**Theorem 4 (C)** *If $P$, $Q$ are true then $P \wedge Q$ is true.*

**Proof.**

| $P$ | $Q$ | $P \Rightarrow (Q \Rightarrow (P \wedge Q))$ | $Q \Rightarrow (P \wedge Q)$ | $P \wedge Q$ |
|---|---|---|---|---|
| H | H | T | M | M |

We can use the above theorems to shorten other proofs.

**Theorem 5 (O)** *If $\sim P, P \vee Q$ are true then $Q$ is true.*

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**Proof.**

| $\sim P$ | $P \vee Q$ | $(\sim P) \wedge (P \vee Q)$ | $(\sim P) \wedge (P \vee Q) \Rightarrow Q$ | $Q$ |
|---|---|---|---|---|
| H | H | C | T | M |

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In all cases the central tautology of the proof is easily verified.

## 1.2 Exercises

For the first two exercises, use a truth table:

1. Verify the tautologies of Theorems E, C, O.
2. Show that $(P \wedge Q) \wedge R$ and $P \wedge (Q \wedge R)$ are equivalent.

The next two exercises are a formalized version of Ex 1.4.4. in the text, and should be proved in a manner similar to Theorems O. You can use Theorems E,C,O to shorten your proofs. In each case only state the hypotheses you need.

Consider the statements

$$\sim P \Rightarrow K, M \Rightarrow P, C \Rightarrow \sim S, \sim S \Rightarrow C, C \vee L, S \vee P$$

and prove the following theorems:

3. If the above statements and $\sim L$ are true then $P$ is true.
4. If the above statements and $\sim M, \sim C$ are true then $P$ is true.