1 Lectures 3 and 4 (1/25 and 2/1/2011)

Key Terms: Open sentence, Universe of discourse; Truth set; Equivalent sentences; Quantified sentences: existential, universal, unique existential; Denials of open sentences; Denial of quantified sentences.

Key Symbols: \( \in, \notin, \subseteq, \supseteq, \land, \lor, \neg, \forall, \exists \)

1.1 Background on set theory

A set is a collection of objects, which are called its elements. If \( R \) and \( S \) are sets, we say \( R \) is a subset of \( S \) if every element of \( R \) is also an element of \( S \).

Definition 1 The sentence \( x \in S \) is a proposition that means "\( x \) is an element of \( S \)"; its negation is denoted \( x \notin S \). The sentence \( R \subseteq S \) is a proposition that means "\( R \) is a subset of \( S \)"; its negation is denoted \( R \nsubseteq S \).

Example 2 Let \( S = \{1, 2, 3\} \), then we have

\[
\begin{array}{c|c|c|c|c}
1 \in S & T & 4 \in S & F & \{1, 2\} \subseteq S & T \\
1 \notin S & F & 4 \notin S & T & \{1, 2\} \nsubseteq S & F \\
\end{array}
\]

If \( S \) is a subset of \( U \) we define the complement of \( S \) in \( U \) to be the set \( U \setminus S \), which consists of all elements of \( U \) that are not in \( S \). If \( U \) is understood from context then we write \( S^c \) instead of \( U \setminus S \).

Now \( (S^c)^c \) consists of those elements of \( U \) that are not in \( S^c \), i.e. those that are not (not in \( S \)), i.e. those that are in \( S \). Therefore we conclude

\[ (S^c)^c = S. \] (1)

Note also that \( S^c \) will contain at least one element unless \( S = U \); thus we get

\[ S^c = \{\} \iff S = U \] (2)

Arguing similarly (or using (1)), we get

\[ S^c = U \iff S = \{\} \] (3)

Example 3 Let \( U = \{1, 2, 3\} \) then we have

\[
\begin{array}{c|c|c|c|c}
S & \{1\} & \{2, 3\} & \{1, 2, 3\} & \{\} \\
S^c & \{2, 3\} & \{1\} & \{\} & \{1, 2, 3\} \\
\end{array}
\]

1.2 Special sets

Some important sets are denoted by special symbols: the empty set \( \emptyset = \{\} \); the natural numbers \( \mathbb{N} = \{1, 2, 3, \ldots\} \); the integers \( \mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, 3, \ldots\} \); the rational numbers \( \mathbb{Q} \); the real numbers \( \mathbb{R} \); and the complex numbers \( \mathbb{C} \).

We recall that a rational number is a real number that can be written as ratio of the form \( p/q \) where \( p \) is an integer and \( q \) is a natural number. A complex number is one that is of the form \( a + bi \) where \( a, b \) are real numbers and \( i = \sqrt{-1} \).

We identify \( p/1 \) with \( p \) and \( a + 0i \) with \( a \), and thus we have

\[ \emptyset \subseteq \mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C} \]
1.3 Open sentences

**Definition 4**  A sentence is said to be open if it contains one or more variables.

For simplicity we first discuss open sentences with a single variable.

**Example 5**  
- a) $x \geq 3$
- b) $P$ is a boy.

**Definition 6**  Let $P(x)$ be an open sentence and let $U$ be a set. We say $U$ is a universe of discourse for $P(x)$, if $P(u)$ is a proposition for every $u \in U$.

A sentence can have several possible universes of discourse. For example, let $P(x)$ be the sentence $x \geq 3$. If we replace $x$ by a natural number or real number, then we get a proposition i.e. a sentence that is either true or false; e.g. $2 \geq 3$ is false, $\pi \geq 3$ is true. However if we replace $x$ by a color, say "red", then we get the meaningless sentence "red" $\geq 3$, which is not a proposition. Thus both $\mathbb{N}$ and $\mathbb{R}$ are possible universes for $x \geq 3$, but the set of colors is not.

**Definition 7**  The truth set of an open sentence $P(x)$ in universe $U$ is the set

$$\{x \in U \mid P(x)\}$$

consisting of all elements of $U$ for which $P(x)$ is true.

<table>
<thead>
<tr>
<th>Universe $U$</th>
<th>${x \in U \mid x \geq 3}$</th>
<th>${x \in U \mid x &lt; 3}$</th>
<th>${x \in U \mid x \leq 2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathbb{N}$</td>
<td>${3, 4, 5, \ldots}$</td>
<td>${1, 2}$</td>
<td>${1, 2}$</td>
</tr>
<tr>
<td>$\mathbb{R}$</td>
<td>$[3, \infty)$</td>
<td>$(-\infty, 3)$</td>
<td>$(-\infty, 2]$</td>
</tr>
</tbody>
</table>

**Definition 9**  Two sentences $Q(x)$ and $P(x)$ are said to be equivalent in $U$ if their truth sets are equal.

**Example 10**  By Example 8 the sentences $x < 3$ and $x \leq 2$ are equivalent in $\mathbb{N}$ but not in $\mathbb{R}$.

**Definition 11**  The negation of an open sentence $P(x)$ is the sen-

**Example 12**  

<table>
<thead>
<tr>
<th>$P(x)$</th>
<th>$\sim P(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x \geq 3$</td>
<td>$x &lt; 3$</td>
</tr>
<tr>
<td>$x$ is a boy</td>
<td>$x$ is not a boy</td>
</tr>
</tbody>
</table>

Let $S$ denote the truth set of $P(x)$ in some universe $U$, and let $R$ denote the truth set of $\sim P(x)$. Then an element $u$ of $U$ belongs to $R$ iff $\sim P(u)$ is true, i.e. iff $P(u)$ is false, i.e. iff $u$ does not belong to $S$. Therefore we conclude that $R = S^c$. In other words we have

$$\{x \in U \mid \sim P(x)\} = \{x \in U \mid P(x)\}^c \quad (4)$$

**Example 13**  Let $P(x)$ be the sentence $x \geq 3$; its negation $\sim P(x)$ is the sen-
tence $x < 3$, and as shown in Example 8, their truth sets are complements of each other in both $\mathbb{N}$ and $\mathbb{R}$.  

2
1.4 Quantifiers

Definition 14 Let \( P(x) \) be an open sentence with universe \( U \). Let \( S \) be the truth set of \( P(x) \) and define three propositions as follows:

\[
\begin{align*}
(\exists x \in U) P(x) & \text{ means } S \neq \emptyset \\
(\forall x \in U) P(x) & \text{ means } S = U \\
(\exists! x \in U) P(x) & \text{ means } S \text{ consists of exactly one element.}
\end{align*}
\]

The symbols \( \exists, \forall, \exists! \) are read "there exists", "for all", "there exists a unique"; and are called existential, universal, and unique existential quantifiers, respectively. The sentences formed using quantifiers in the above definition are called quantified sentences, and we emphasise again that each is a proposition, whose truth value can be determined by examining the truth set of \( P(x) \).

If \( U \) is understood from context, then we simply write \( (\exists x) P(x) \), \( (\forall x) P(x) \), \( (\exists! x) P(x) \) for the three quantified propositions.

Example 15 Let \( U = \mathbb{N} \), then we have the following truth table

<table>
<thead>
<tr>
<th>( P(x) )</th>
<th>Truth Set</th>
<th>( (\exists x) P(x) )</th>
<th>( (\forall x) P(x) )</th>
<th>( (\exists! x) P(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x \geq 3 )</td>
<td>{3, 4, 5, \cdots}</td>
<td>( T )</td>
<td>( F )</td>
<td>( F )</td>
</tr>
<tr>
<td>( x &lt; 2 )</td>
<td>{1}</td>
<td>( T )</td>
<td>( F )</td>
<td>( T )</td>
</tr>
<tr>
<td>( x &gt; 0 )</td>
<td>( \mathbb{N} )</td>
<td>( T )</td>
<td>( T )</td>
<td>( F )</td>
</tr>
<tr>
<td>( x &lt; 0 )</td>
<td>( \emptyset )</td>
<td>( F )</td>
<td>( F )</td>
<td>( F )</td>
</tr>
</tbody>
</table>

Let \( S \) be the truth set of \( P(x) \) in the universe \( U \). Now by formula (2) we have

\[
S = \emptyset \iff S^c = U
\]

Since \( S \neq \emptyset \) means \( (\exists x \in U) P(x) \), the left side means \( \neg(\exists x \in U) P(x) \). Also by (4) the truth set of \( \neg P(x) \) is \( S^c \), therefore the right side means \( (\forall x \in U)(\neg P(x)) \). Thus we conclude

\[
\neg(\exists x \in U) P(x) \iff (\forall x \in U)(\neg P(x)) \quad (5)
\]

Similarly one can show that

\[
\neg(\forall x \in U) P(x) \iff (\exists x \in U)(\neg P(x)) \quad (6)
\]

What we have just done is given an "informal proof" of a theorem. In the next lecture we will see how to do formal proofs.

1.5 Exercises

1. Give a proof of (3) similar to that (2).
2. Give a proof deducing (3) from (1) and (2).
3. Give a proof of (6) similar to that (5).
4. Give an example of an open sentence \( P(x) \) with universe \( U \) such that \( (\exists x \in U) P(x) \) is false but \( (\forall x \in U) P(x) \) is true.