1 Lecture 10 (2/22/2011)

Key Terms: More proofs with quantifiers – $\exists x$ and $\forall x$

1.1 Proofs of $\exists x \, P(x)$

The quantified proposition $\exists x \, P(x)$ is the statement that

$\text{There is some element } u \text{ in the universe } U \text{ for which } P(u) \text{ is true.}$

How do we prove such a statement? The most direct way is to exhibit such an element. The proof usually starts with a statement of the form "Define $u = ...$", as is shown in the following example.

**Theorem 1**  There exists a natural number that is both even and prime.

$[\exists n \, (n \text{ is even}) \land (n \text{ is prime}); \ U = \mathbb{N}]$

**Proof.**

1. Define $u = 2$.
2. Then $u$ is a natural number.
3. Since $u = 2(1)$, $u$ is even.
4. Since the only factors of $u$ are 1, 2, $u$ is prime.
5. Therefore $u$ is both even and prime.
6. Thus there exists a natural number that is both even and prime.

**Remark 3**  The main point of the above example is that it is not always "easy" to come up with the desired element(s) that establish the truth of $\exists x \, P(x)$. An alternative strategy that sometimes works is to argue by contradiction, namely to assume $\neg(\exists x \, P(x))$ and derive a false statement. In this context it is useful to remember that $\neg(\exists x \, P(x))$ is equivalent to $\forall x \, (\neg P(x))$ (1)
1.2 Proofs of $\exists! x \ P(x)$

The proposition $\exists! x \ P(x)$ is the statement that

*There is exactly one element $u$ in the universe $U$ for which $P(u)$ is true.*

The most direct way to prove this statement requires two steps:

1. **Existence:** Find an element $u$ in the universe such that $P(u)$ is true.
2. **Uniqueness:** Show that $u$ is the *only* such element, by proving that if $P(v)$ is true then $v = u$.

Here is an example:

**Theorem 4** There exists a unique natural number that is both even and prime.

**Proof.** (Existence) By Theorem 1 the natural number 2 is both even and prime.

(Uniqueness) Let $x$ be a natural number. Suppose $x$ is both even and prime. Since $x$ is even, 2 divides $x$. Since $x$ is prime, the only divisors of $x$ are 1 and $x$. Therefore $2 = 1$ or $2 = x$. Since $2 \neq 1$, we must have $2 = x$. ■

1.3 Exercises

1. The pigeonhole principle says: "If $m$ letters have been distributed into $n$ pigeonholes, and $m > n$, then there exists a pigeonhole with 2 or more letters." Give a careful proof of this, following the strategy of Remark 3.

   [Hint: Here the universe is the set of pigeonholes; $P(x)$ is the property that pigeonhole $x$ contains 2 or more letters; $\sim P(x)$ is the property that $x$ contains 0 or 1 letters. By (1) the denial of $\exists x P(x)$ is $\forall x (\sim P(x))$. What relation between $m$ and $n$ does this imply? Where’s the contradiction?]

2. If $a, b, c, d$ are natural numbers with $\frac{a}{b} < \frac{c}{d}$ then there is a natural number $n$ such that $\frac{a}{b} < \frac{d}{b+nd} < \frac{c}{d}$

   [Hint: Of course $n$ will depend in some manner on $a, b, c, d$ and you have to "guess" which $n$ will work, this is the creative part! For the more mundane part, to prove an inequality of the form $\frac{x}{y} < \frac{u}{v}$ with $x, y, u, v$ positive, you simply need to show that $xv < yu$.]

3. In the notation of Exercise 2, suppose further that $bc - ad = 1$ and then prove that $n$ is unique.

   [Hint: Show that the natural numbers bigger and smaller than the $n$ you found in Exercise 2, will not work.]