Math 356 Notes

Notes on the 9/06 and 9/10 lectures:

We recall that $\mathbf{Z}_{\geq k}$ means the set of integers greater than or equal to k.

Least integer principle (LIP): Every non-empty subset of $\mathbf{Z}_{\geq k}$ has a least element.

Principle of mathematical induction (PMI): Suppose *P(n)* is an assertion about integers *n* in $\mathbf{Z}_{\geq k}$. If (1) *P(k)* is true and (2) *P(m)* implies *P(m+1)* for each *m*, then *P(n)* is true for all *n* in $\mathbf{Z}_{\geq k}$.

Principle of strong mathematical induction (PSI): Suppose *P(n)* is an assertion about integers *n* in $\mathbf{Z}_{\geq k}$. If (1) *P(k)* is true and (2') *{P(k),P(k+1),…,P(m)}* together imply *P(m+1)* for each *m*., then *P(n)* is true for all *n* in $\mathbf{Z}_{\geq k}$.

As we shall see below, the three principles are equivalent.

Theorem: *LIP implies PMI.*

Proof: Suppose *P(n)* satisfies (1) and (2) of PMI, and let *S* be the set of integers in $\mathbf{Z}_{\geq\backslash k}$ for which *P(n)* is false. If *S* is non-empty; then by LIP it has a least element *d,* which is strictly larger than *k* by (1). Therefore *P(d-1)* must be true, and by (2) *P(d)* must be true as well, which is a contradiction. Therefore S must be empty, and so *P(n)* is true for all *n.*

Theorem: *PMI implies PSI.*

Proof: Suppose *P(n)* satisfies (1) and (2') of PSI. We want to deduce that P(n) is true for all $n \geq k$. Let *Q(n)* be the assertion that "{*P(k), P(k+1), …, P(n)*} are all true". Then *Q(n)* satisfies (1) and (2) of PMI, and so *Q(n)* is true for all $n \geq k$. Hence *P(n)* is true for all $n \geq k$.

Theorem: *PSI implies LIP.*

Proof: Let *S* be a subset of $\mathbf{Z}_{\geq k}$ that has no least element; we must show that *S* is empty. Let *P(n)* be the assertion "*n does not belong to S* ". Since *S* has no least element, *k* cannot belong to *S*, therefore *P(k)* is true. Now suppose *{P(k),P(k+1),…,P(m)}* are all true, *i.e. k, k+1, …, m* do not belong to *S*; then *m+1* cannot belong to *S* either, else it would be the least element of *S*. By PSI, *P(n)* is true for all *n* in $\mathbf{Z}_{\geq k}$, and so *S* must be empty.