1 Onto and injective relations

Let $A, B$ be sets and let $R \subseteq A \times B$ be a relation.

**Definition 1** We say that $R$ is onto if $\text{Rng}(R) = B$.

**Example 2** Let $A = \{a, b, c\}$ and let $B = \{1, 2, 3, 4\}$ and consider the following relations

\[
R_1 = \{(a, 1), (b, 2)\}, \quad R_2 = \{(a, 1), (a, 2), (b, 3), (c, 4)\}
\]

\[
R_3 = \{(a, 1), (b, 2), (c, 2)\}, \quad R_4 = \{(b, 1), (b, 3), (c, 2)\}
\]

Then $\text{Rng}(R_1) = \text{Rng}(R_3) = \{1, 2\}$, $\text{Rng}(R_2) = \{1, 2, 3\}$, therefore $R_1, R_3, R_4$ are not onto. However $\text{Rng}(R_2) = \{1, 2, 3, 4\} = B$, therefore $R_2$ is onto.

**Definition 3** We say that $R$ is injective if
\[
\forall a, b, b' [(a, b) \in R \wedge (a, b') \in R \implies b = b']
\]

**Example 4** In the example above $R_2$ is not injective because $(a, 1) \in R_2, (a, 2) \in R_2$ but $1 \neq 2$. Similarly $R_4$ is not injective $(b, 1), (b, 3) \in R_4$, but $1 \neq 3$. However $R_1, R_3$ are both injective.

**Theorem 5** If $R \subseteq A \times B$ and $S \subseteq B \times C$ are both injective, then so is $S \circ R$.

**Proof.** Let $A, B, C$ be sets and let $R \subseteq A \times B, S \subseteq B \times C$ be relations. Assume that $R, S$ are both injective.

Since $R \subseteq A \times B$ and $S \subseteq B \times C$, we have $S \circ R \subseteq A \times C$.

Let $a \in A$ and $c, c' \in C$, and suppose $(a, c) \in S \circ R \wedge (a, c') \in S \circ R$.

Since $(a, c) \in S \circ R$, there exists $b \in B$ such that $(a, b) \in R$ and $(b, c) \in S$.

Since $(a, c') \in S \circ R$, there exists $b' \in B$ such that $(a, b') \in R$ and $(b', c') \in S$.

Since $(a, b) \in R \wedge (a, b') \in R$, and $R$ is injective we get $b = b'$.

Thus we have $(b, c) \in S$ and, by substitution, $(b, c') \in S$.

Since $S$ is injective, we get $c = c'$.

Therefore $S \circ R$ is injective.

**Theorem 6** If $R \subseteq A \times B$ and $S \subseteq B \times C$ are both onto, then so is $S \circ R$.

**Proof.** Exercise.
2 More on functions

Recall that $R \subseteq A \times B$ is said to be a function if for each $a \in A$ there is a unique $b \in B$ such that $(a, b) \in R$. We can also reformulate this as follows:

**Definition 7** $R \subseteq A \times B$ is a function if

1. $\text{Dom}(R) = A$
2. $\forall a, a', b [(a, b) \in R \land (a', b) \in R \implies a = a']$

Note that conditions 1 and 2 above are quite similar to the definitions of "onto" and "injective". In fact we have the following result.

**Theorem 8** If $R \subseteq A \times B$, then $R$ is a function iff $R^{-1}$ is onto and injective.

**Proof.** Exercise. ■

We can combine the previous three theorems to prove an important result.

**Theorem 9** If $R \subseteq A \times B$ and $S \subseteq B \times C$ are functions, then so is $S \circ R$.

**Proof.** Suppose $R \subseteq A \times B$ and $S \subseteq B \times C$ are functions. Then by Theorem 8 $R^{-1}$ and $S^{-1}$ are onto and injective. By Theorems 5 and 6 $R^{-1} \circ S^{-1}$ is onto and injective. But $R^{-1} \circ S^{-1} = (S \circ R)^{-1}$. Therefore by Theorem 8 $S \circ R$ is a function. ■

2.1 Bijections

**Definition 10** A relation $R \subseteq A \times B$ is said to be a bijection, if $R$ is a function that is both injective and onto.

**Theorem 11** $R \subseteq A \times B$ is a bijection iff $R^{-1}$ is a bijection.

**Proof.** Exercise. ■

**Theorem 12** Let $S$ be a collection of sets, (i.e. elements of $S$ are themselves sets). Define a relation on $S$ as follows

$\mathcal{R} = \{(A, A') \in S \times S \mid \exists R \subseteq A \times A'$ such that $R$ is a bijection$\}$

Then $\mathcal{R}$ is an equivalence relation on $S$.

**Proof.** Exercise. ■

2.2 Exercises

2. Prove Theorem 8.
3. Prove Theorem 11.
4. Prove Theorem 12.