Math 300, Lecture 24, Siddhartha Sahi

1 Functions

Let A, B be sets and let $R \subseteq A \times B$ be a relation.

Definition 1 *We say that $R \subseteq A \times B$ is a function if the following two conditions hold:*

1. *$Dom\,(R) = A$.*
2. *for all $a \in Dom\,(R)$ there is a unique $b \in B$ such that $(a, b) \in R$.*

These two conditions can be combined into a single condition.

Theorem 2 *If $R \subseteq A \times B$, then R is a function iff*

$$\forall a \in A \ \ (\exists! b \in B \ \ [(a, b) \in R])$$

Proof. Exercise. ■

Example 3 *Let $A = \{a, b, c\}$ and let $B = \{1, 2, 3, 4\}$ and consider the following relations*

$$R_1 = \{(a, 1), (b, 2)\}, R_2 = \{(a, 1), (a, 2), (b, 3), (c, 4)\}$$
$$R_3 = \{(a, 1), (b, 2), (c, 2)\}, R_4 = \{(a, 1), (b, 3), (c, 2)\}$$

Then R_1 and R_2 are not functions, because they fail to satisfy conditions 1 and 2 respectively. However R_3 and R_4 are functions.

2 Functions on $\mathbb{R}$

If R is a relation on an infinite set such as $\mathbb{R}$ (the real numbers). Then it is impractical to list all ordered pairs that make up R. However we can give some kind of a formula $f(x)$ such that

$$R = \{(x, f(x)) \mid x \in \mathbb{R}\} \subseteq \mathbb{R} \times \mathbb{R}$$

This is the point of view followed in calculus, where the formula f is considered to be the "function".

On the other hand the set R, which we are considering as the actual function, is called the "graph" of f in calculus. Note that $\mathbb{R} \times \mathbb{R}$ can be identified with the coordinate plane, and R can be drawn as a certain set of points in the plane. For example for $f(x) = x^2$, the graph of f will be the familiar parabola.

More generally, any set of points in the plane defines a subset of $\mathbb{R} \times \mathbb{R}$ and hence a relation on $\mathbb{R}$.

Theorem 4 *If S is a subset of $\mathbb{R} \times \mathbb{R}$, then S is a function if and only if each vertical line of the form $x =$constant meets S in exactly one point.*

Proof. Exercise ■

2.1 Exercises

1. Prove Theorem 2.

2. Prove Theorem 4.