1 Functions

Let $A, B$ be sets and let $R \subseteq A \times B$ be a relation.

**Definition 1** We say that $R \subseteq A \times B$ is a function if the following two conditions hold:

1. $\text{Dom}(R) = A$.
2. for all $a \in \text{Dom}(R)$ there is a unique $b \in B$ such that $(a, b) \in R$.

These two conditions can be combined into a single condition.

**Theorem 2** If $R \subseteq A \times B$, then $R$ is a function iff

$$\forall a \in A \ (\exists! b \in B \ [(a, b) \in R])$$

**Proof.** Exercise. $lacksquare$

**Example 3** Let $A = \{a, b, c\}$ and let $B = \{1, 2, 3, 4\}$ and consider the following relations

- $R_1 = \{(a, 1), (b, 2)\}$, $R_2 = \{(a, 1), (a, 2), (b, 3), (c, 4)\}$
- $R_3 = \{(a, 1), (b, 2), (c, 2)\}$, $R_4 = \{(a, 1), (b, 3), (c, 2)\}$

Then $R_1$ and $R_2$ are not functions, because they fail to satisfy conditions 1 and 2 respectively. However $R_3$ and $R_4$ are functions.

2 Functions on $\mathbb{R}$

If $R$ is a relation on an infinite set such as $\mathbb{R}$ (the real numbers). Then it is impractical to list all ordered pairs that make up $R$. However we can give some kind of a formula $f(x)$ such that

$$R = \{(x, f(x)) \mid x \in \mathbb{R}\} \subseteq \mathbb{R} \times \mathbb{R}$$

This is the point of view followed in calculus, where the formula $f$ is considered to be the "function".
On the other hand the set $R$, which we are considering as the actual function, is called the "graph" of $f$ in calculus. Note that $\mathbb{R} \times \mathbb{R}$ can be identified with the coordinate plane, and $R$ can be drawn as a certain set of points in the plane. For example for $f(x) = x^2$, the graph of $f$ will be the familiar parabola.

More generally, any set of points in the plane defines a subset of $\mathbb{R} \times \mathbb{R}$ and hence a relation on $\mathbb{R}$.

**Theorem 4** If $S$ is a subset of $\mathbb{R} \times \mathbb{R}$, then $S$ is a function if and only if each vertical line of the form $x = \text{constant}$ meets $S$ in exactly one point.

**Proof.** Exercise ■

### 2.1 Exercises

1. Prove Theorem 2.