1 Ordered pairs and Cartesian product

1.1 Ordered pairs

The Cartesian coordinate system on the plane assigns to each point a pair of real numbers \((a, b)\). In this situation the order matters; \((1, 2)\) and \((2, 1)\) are not the same point. By contrast, \(\{1, 2\}\) and \(\{2, 1\}\) both represent the same set. How then are we define an ordered pair?

We can do this in some generality. Let us fix a universe of discourse \(U\). If \(a, b \in U\), then we can consider the following sets:

\[
\{a\}, \{b\}, \{a, b\} = \{a\} \cup \{b\}
\]

In the above we do not require \(a\) and \(b\) to be different. However note that

\[
\{a, a\} = \{a\} \cup \{a\} = \{a\}.
\] (1)

**Definition 1** For \(a, b \in U\), we define the ordered pair \((a, b)\) to be the set

\[
(a, b) = \{\{a\}, \{a, b\}\}
\]

In other words \((a, b)\) is a set whose elements are the sets \(\{a\}\) and \(\{a, b\}\). Note that by formula (1) we get

\[
(a, a) = \{\{a\}, \{a, a\}\} = \{\{a\}, \{a\}\} = \{\{a\}\}
\] (2)

The main property of ordered pairs is contained in the following theorem.

**Theorem 2** If \((a, b) = (c, d)\) then \(a = c\) and \(b = d\).

**Proof.** Let \(a, b, c, d \in U\) and suppose \((a, b) = (c, d)\). Then by definition

\[
\{\{a\}, \{a, b\}\} = \{\{c\}, \{c, d\}\}
\] (3)

Since \(\{a\}\) is an element of the left set, therefore by definition of equality of sets, \(\{a\}\) must be an element of the right set as well. This means that either \(\{a\} = \{c\}\) or \(\{a\} = \{c, d\}\). First suppose that \(\{a\} = \{c\}\), then since \(c\) belongs to \(\{c\}\), \(c\) belongs to \(\{a\}\) and hence \(a = c\). Next suppose \(\{a\} = \{c, d\}\), since \(c\) belongs to \(\{c, d\}\), once again \(c\) belongs to \(\{a\}\) and hence \(a = c\). Therefore we conclude \(a = c\) in either case.
We now prove \( b = d \). Since we already know \( a = c \), formula (3) becomes

\[
\{\{a\}, \{a, b\}\} = \{\{a\}, \{a, d\}\}
\]  (4)

We consider two cases. First suppose that \( b \neq a \). Now since \( \{a, b\} \) is an element of the left set, \( \{a, b\} \) is an element of the right set. This means either \( \{a, b\} = \{a\} \) or \( \{a, b\} = \{a, d\} \). Now \( \{a, b\} = \{a\} \) would give \( b \in \{a\} \), and hence \( b = a \), which is a contradiction. Therefore we must have \( \{a, b\} = \{a, d\} \), which implies \( b \in \{a, d\} \). Since \( b \neq a \) it follows that \( b = d \).

To complete the proof we must deal with case \( b = a \). In this case by (2) formula (4) becomes

\[
\{\{a\}\} = \{\{a\}, \{a, d\}\}
\]  (5)

We leave it as an exercise to show that (5) implies \( a = d \), and hence \( b = d \).

1.2 Cartesian product

**Definition 3** Suppose \( A \) and \( B \) are sets. We define the Cartesian product \( A \times B \) to be the set of all ordered pairs whose first element is from \( A \) and second element from \( B \). Thus we have

\[
A \times B = \{(a, b) : a \in A, b \in B\}
\]

Equivalently, if \( (x, y) \) is an ordered pair then

\[
(x, y) \in A \times B \iff (x \in A) \land (y \in B)
\]  (6)

**Example 4** Let \( A = \{1, 2\}, B = \{2, 3\} \) then we have

\[
A \times B = \{(1, 2), (1, 3), (2, 2), (2, 3)\}
\]

**Theorem 5** Suppose \( A, B, C, D \) are sets, then

1. \( (A \times B) \cap (A \times C) = A \times (B \cap C) \)
2. \( (A \times B) \cup (A \times C) = A \times (B \cup C) \)
3. \( (A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D) \)
4. \( (A \times B) \cup (A \times C) \subseteq (A \cup C) \times (B \cup D) \)

We prove 2) and 4) leaving 1) and 3) as exercises.

**Remark 6** If \( S \) and \( T \) are sets of ordered pairs, then to prove \( S = T \) we show that for all ordered pairs \( (x, y) \) we have \( (x, y) \in S \iff (x, y) \in T \).
**Proof.** (of Theorem 5 part 2) Let \( A, B, C \) be sets, and let \((x, y)\) be an ordered pair. Then we have

\[
(x, y) \in (A \times B) \cup (A \times C)
\]

\[
\iff [(x, y) \in A \times B] \lor [(x, y) \in A \times C] \quad \text{(by definition of \( \cup \))}
\]

\[
\iff (x \in A \land y \in B) \lor (x \in A \land y \in C) \quad \text{(by (6))}
\]

\[
\iff (x \in A) \land (y \in B \lor y \in C) \quad \text{(by (7))}
\]

\[
\iff (x, y) \in A \times (B \cup C) \quad \text{(by (6))}
\]

Therefore \((A \times B) \cup (A \times C) = A \times (B \cup C)\). ■

**Remark 7** We leave it as an exercise to verify the equivalence

\[(P \land Q) \lor (P \land R) \iff P \land (Q \lor R) \quad (7)\]

**Proof.** (of Theorem 5 part 4) Let \( A, B, C, D \) be sets, and let \((x, y)\) be an ordered pair. Then we have

\[
(x, y) \in (A \times B) \cup (A \times C)
\]

\[
\iff [(x, y) \in A \times B] \lor [(x, y) \in A \times C] \quad \text{(by definition of \( \cup \))}
\]

\[
\iff [x \in A \land y \in B] \lor [x \in C \land y \in D] \quad \text{(by (6))}
\]

\[
\iff [x \in A \lor x \in C] \land [y \in B \lor y \in D] \quad \text{(by (8))}
\]

Therefore \((A \times B) \cup (A \times C) \subseteq (A \cup C) \times (B \cup D)\) ■

**Remark 8** We leave it as an exercise to verify the implication

\[(P \land Q) \lor (R \land S) \Rightarrow (P \lor R) \land (Q \lor S) \quad (8)\]

### 1.3 Exercises

1. Finish the proof of Theorem 2 by showing:
   If \(\{a\} = \{a, b\}\), then \(a = b\).

2. Use a truth table to verify (7) and (8).

3. Prove parts 1) and 3) of Theorem 5 by using appropriate equivalences.
   [Hint: These equivalence look very similar to the theorems themselves!]

4. Use a truth table to verify the equivalences used in Exercise 3.