Math 300, Lecture 15, Siddhartha Sahi

1 Ordered pairs and Cartesian product

1.1 Ordered pairs

The Cartesian coordinate system on the plane assigns to each point a pair of real numbers (a, b). In this situation the *order* matters; $(1, 2)$ and $(2, 1)$ are not the same point. By contrast, $\{1, 2\}$ and $\{2, 1\}$ both represent the same set. How then are we define an ordered pair?

We can do this in some generality. Let us fix a universe of discourse U. If $a, b \in U$, then we can consider the following sets:

$$\{a\}, \{b\}, \{a, b\} = \{a\} \cup \{b\}$$

In the above we do not require a and b to be different. However note that

$$\{a, a\} = \{a\} \cup \{a\} = \{a\}. \tag{1}$$

Definition 1 *For $a, b \in U$, we define the* ordered pair (a, b) *to be the set*

$$(a, b) = \{\{a\}, \{a, b\}\}$$

In other words (a, b) is a set whose elements are the sets $\{a\}$ and $\{a, b\}$. Note that by formula (1) we get

$$(a, a) = \{\{a\}, \{a, a\}\} = \{\{a\}, \{a\}\} = \{\{a\}\} \tag{2}$$

The main property of ordered pairs is contained in the following theorem.

Theorem 2 *If $(a, b) = (c, d)$ then $a = c$ and $b = d$.*

Proof. Let $a, b, c, d \in U$ and suppose $(a, b) = (c, d)$. Then by definition

$$\{\{a\}, \{a, b\}\} = \{\{c\}, \{c, d\}\} \tag{3}$$

Since $\{a\}$ is an element of the left set, therefore by definition of equality of sets, $\{a\}$ must be an element of the right set as well. This means that either $\{a\} = \{c\}$ or $\{a\} = \{c, d\}$. First suppose that $\{a\} = \{c\}$, then since c belongs to $\{c\}$, c belongs to $\{a\}$ and hence $a = c$. Next suppose $\{a\} = \{c, d\}$, since c belongs to $\{c, d\}$, once again c belongs to $\{a\}$ and hence $a = c$. Therefore we conclude $a = c$ in either case.

We now prove $b = d$. Since we already know $a = c$, formula (3) becomes

$$\{\{a\}, \{a, b\}\} = \{\{a\}, \{a, d\}\} \tag{4}$$

We consider two cases. First suppose that $b \neq a$. Now since $\{a, b\}$ is an element of the left set, $\{a, b\}$ is an element of the right set. This means either $\{a, b\} = \{a\}$ or $\{a, b\} = \{a, d\}$. Now $\{a, b\} = \{a\}$ would give $b \in \{a\}$, and hence $b = a$, which is a contradiction. Therefore we must have $\{a, b\} = \{a, d\}$, which implies $b \in \{a, d\}$. Since $b \neq a$ it follows that $b = d$.

To complete the proof we must deal with case $b = a$. In this case by (2) formula (4) becomes

$$\{\{a\}\} = \{\{a\}, \{a, d\}\} \tag{5}$$

We leave it as an exercise to show that (5) implies $a = d$, and hence $b = d$. ■

1.2 Cartesian product

Definition 3 *Suppose A and B are sets. We define the Cartesian product $A \times B$ to be the set of all ordered pairs whose first element is from A and second element from B. Thus we have*

$$A \times B = \{(a, b) : a \in A, b \in B\}$$

Equivalently, if (x, y) is an ordered pair then

$$(x, y) \in A \times B \;\Leftrightarrow (x \in A) \wedge (y \in B) \tag{6}$$

Example 4 *Let $A = \{1, 2\}, B = \{2, 3\}$ then we have*

$$A \times B = \{(1, 2), (1, 3), (2, 2), (2, 3)\}$$

Theorem 5 *Suppose A, B, C, D are sets, then*

1. $(A \times B) \cap (A \times C) = A \times (B \cap C)$
2. $(A \times B) \cup (A \times C) = A \times (B \cup C)$
3. $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$
4. $(A \times B) \cup (A \times C) \subseteq (A \cup C) \times (B \cup D)$

We prove 2) and 4) leaving 1) and 3) as exercises.

Remark 6 *If S and T are sets of ordered pairs, then to prove $\boxed{S = T}$ we show that for all ordered pairs (x, y) we have $\boxed{(x, y) \in S \Leftrightarrow (x, y) \in T}$.*

Proof. (of Theorem 5 part 2) Let A, B, C be sets, and let (x, y) be an ordered pair. Then we have

$$\begin{aligned}
&(x,y) \in (A \times B) \cup (A \times C) \\
&\Leftrightarrow [(x,y) \in A \times B] \vee [(x,y) \in A \times C] \quad \text{(by definition of } \cup\text{)} \\
&\Leftrightarrow (x \in A \wedge y \in B) \vee (x \in A \wedge y \in C) \quad \text{(by (6))} \\
&\Leftrightarrow (x \in A) \wedge (y \in B \vee y \in C) \quad \text{(by (7))} \\
&\Leftrightarrow (x,y) \in A \times (B \cup C) \quad \text{(by (6))}
\end{aligned}$$

Therefore $(A \times B) \cup (A \times C) = A \times (B \cup C)$. ■

Remark 7 *We leave it is an exrecise to verify the equivalence*

$$(P \wedge Q) \vee (P \wedge R) \Leftrightarrow P \wedge (Q \vee R) \tag{7}$$

Proof. (of Theorem 5 part 4) Let A, B, C, D be sets, and let (x, y) be an ordered pair. Then we have

$$\begin{aligned}
&(x,y) \in (A \times B) \cup (A \times C) \\
&\Leftrightarrow [(x,y) \in A \times B] \vee [(x,y) \in A \times C] \quad \text{(by definition of } \cup\text{)} \\
&\Leftrightarrow [x \in A \wedge y \in B] \vee [x \in C \wedge y \in D] \quad \text{(by (6))} \\
&\Rightarrow [x \in A \vee x \in C] \wedge [y \in B \vee y \in D] \quad \text{(by (8))}
\end{aligned}$$

Therefore $(A \times B) \cup (A \times C) \subseteq (A \cup C) \times (B \cup D)$ ■

Remark 8 *We leave it is an exercise to verify the implication*

$$(P \wedge Q) \vee (R \wedge S) \Rightarrow (P \vee R) \wedge (Q \vee S)\,. \tag{8}$$

1.3 Exercises

1. Finish the proof of Theorem 2 by showing:
   If $\{\{a\}\} = \{\{a\}, \{a, d\}\}$ then $a = d$.
2. Use a truth table to verify (7) and (8)
3. Prove parts 1) and 3) of Theorem 5 by using appropriate equivalences.
   [Hint: These equivalence look very similar to the theorems themselves!]
4. Use a truth table to verify the equivalences used in Exercise 3.