Math 300, Lecture 13-14, Siddhartha Sahi

1 Set Theory

1.1 Open sentences and sets

If $P(x)$ is an open sentence with universe U, then in lectures 3-4 we considered the truth set

$$\{x \in U \mid P(x)\},$$

which is a subset of the universe U.

Conversely if S is a subset of the universe we can consider the open sentence

$$x \in S,$$

which is true if x is an element of S and false otherwise.

Example 1 *Suppose the universe is* $U = \{1, 2, 3, 4\}$.

1. *Let* $P(x)$ *be the sentence* $x \geq 3$*; the truth set of* $P(x)$ *is* $\{3, 4\}$.
2. *Let* $S = \{1, 3\}$*; the sentence* $x \in S$ *is true if* $x = 1, 3$ *and false if* $x = 2, 4$.

Most theorems about the set S can be formulated in terms of the open sentence $x \in S$. This allows us to prove such theorems by using the proof techniques that we discussed in earlier lectures. Two important notions involve subsets and set equality.

1.2 Subsets

Definition 2 *Suppose* S *and* T *are sets.*

1. *We say that* S *is a subset of* T*, written* $S \subseteq T$*, if* $\forall x\ (x \in S \Rightarrow x \in T)$.
2. *We say that* $S = T$ *if* $(S \subseteq T)$ *and* $(T \subseteq S)$*; or equivalently*
   (a) $\forall x\ (x \in S \Leftrightarrow x \in T)$
   (b) $\forall x\ (x \in S \Rightarrow x \in T) \wedge \forall x\ (x \in T \Rightarrow x \in S)$

Theorem 3 *Suppose* A, B, C *are sets. If* $A \subseteq B$ *and* $B \subseteq C$ *then* $A \subseteq C$.

Proof.

1. Let A, B, C be sets.
2. Suppose $A \subseteq B$ and $B \subseteq C$.
3. Suppose $x \in A$.
4. Since $A \subseteq B$, we deduce $x \in B$.
5. Since $B \subseteq C$, we deduce $x \in C$.
6. Thus $A \subseteq C$.

■

Remark 4 *Note that lines 1-2 are a restatement of the hypotheses; while lines 3-6 prove $A \subseteq C$, by establishing $\forall x \ (x \in A \Rightarrow x \in C)$.*

Definition 5 *If n is an integer, we define $n\mathbb{Z}$ to be the set*

$$n\mathbb{Z} = \{\cdots, -2n, -n, 0, n, 2n, 3n, \cdots\}$$

Thus $n\mathbb{Z}$ consists of all integers which are multiples of n.

Theorem 6 *If n is a multiple of m then $n\mathbb{Z}$ is a subset of $m\mathbb{Z}$.*

Proof. Let m, n be integers. Suppose n is a multiple of m. Then there is an integer k such that $n = mk$.

Now suppose $l \in n\mathbb{Z}$. Then l is a multiple of n. Thus there is an integer p such that $l = np$. Therefore we get $l = mkp = m\,(kp)$. Since kp is an integer, we deduce that l is a multiple of m. Therefore $l \in m\mathbb{Z}$.

Thus $n\mathbb{Z}$ is a subset of $m\mathbb{Z}$. ■

We can also prove the converse of the previous theorem.

Theorem 7 *If $n\mathbb{Z}$ is a subset of $m\mathbb{Z}$ then n is a multiple of m.*

Proof. Let m, n be integers, and suppose $n\mathbb{Z}$ is a subset of $m\mathbb{Z}$. Since $n = n \cdot 1$, n is a multiple of n. By definition of $n\mathbb{Z}$, $n \in n\mathbb{Z}$. Since $n\mathbb{Z}$ is a subset of $m\mathbb{Z}$, we deduce $n \in m\mathbb{Z}$. Hence by definition of $m\mathbb{Z}$, n is a multiple of m. ■

1.3 Power set

Definition 8 *If S is a set, we define the* power set *of S, written $\mathcal{P}(S)$, to be the set whose* elements *are all the* subsets *of S.*

Example 9 *If $S = \{1, 2\}$ then $\mathcal{P}(S)$ has 4 elements*

$$\mathcal{P}(S) = \{\emptyset, \{1\}, \{2\}, S\}$$

Theorem 10 *If $S \subseteq T$ then $\mathcal{P}(S) \subseteq \mathcal{P}(T)$.*

Proof. Let S, T be sets, and suppose $S \subseteq T$.

Let A be an element of $\mathcal{P}(S)$. Then A is a subset of S. Since S is a subset of T, by Theorem 3, A is a subset of T. Therefore A is an element of $\mathcal{P}(T)$.

Thus we conclude that $\mathcal{P}(S) \subseteq \mathcal{P}(T)$. ■

The converse of this theorem holds as well.

Theorem 11 *If $\mathcal{P}(S) \subseteq \mathcal{P}(T)$ then $S \subseteq T$.*

Proof. Let S, T be sets, and suppose $\mathcal{P}(S) \subseteq \mathcal{P}(T)$.

Since S is a subset of S, it follows that S is an element of $\mathcal{P}(S)$. Since $\mathcal{P}(S)$ is a subset of $\mathcal{P}(T)$ we deduce that S is an element of $\mathcal{P}(T)$. By definition of $\mathcal{P}(T)$ it follows that $S \subseteq T$. ■

We can also prove equality for power sets.

Theorem 12 *If $\mathcal{P}(S) = \mathcal{P}(T)$ then $S = T$.*

Proof. Let S, T be sets and suppose $\mathcal{P}(S) = \mathcal{P}(T)$. Then $\mathcal{P}(S) \subseteq \mathcal{P}(T)$ and $\mathcal{P}(T) \subseteq \mathcal{P}(S)$. By the previous theorem we have $S \subseteq T$ and $T \subseteq S$. It follows that $S = T$. ■

1.4 Set Operations

We now discuss set operations, i.e. ways of combining sets to get new sets.

Definition 13 *If A and B are sets, we define*

1. *(Union) $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$*
2. *(Intersection) $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$*
3. *(Difference) $A \setminus B = \{x \mid x \in A \text{ and } x \notin B\}$.*

Theorems about intersections and unions can also be proved by converting them to theorems about $x \in A$. For example if P, Q, R are three propositions then we have the following tautology

$$P \wedge (Q \vee R) \iff (P \wedge Q) \vee (P \wedge R) \tag{1}$$

[You should verify this with a truth table if you forgot!] This implies the following result.

Theorem 14 *If A, B, C are sets then $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$*

Proof.

1. Let A, B, C be sets.
2. Let x be an element of $A \cap (B \cup C)$.
3. Then $(x \in A) \wedge [(x \in B) \vee (x \in C)]$

4. By (1) $[(x \in A) \wedge (x \in B)] \vee [(x \in A) \wedge (x \in C)]$
5. Thus $x \in (A \cap B) \cup (A \cap C)$.
6. Therefore $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$
7. Now let x be an element of $(A \cap B) \cup (A \cap C)$.
8. Then $[(x \in A) \wedge (x \in B)] \vee [(x \in A) \wedge (x \in C)]$.
9. By (1) $(x \in A) \wedge [(x \in B) \vee (x \in C)]$
10. Thus $x \in A \cap (B \cup C)$
11. Therefore $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$.
12. By lines 6, 11 we get $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

■

We can give this proof more succinctly in the "iff" form.
Proof. Let A, B, C be sets. Then we have

$$\begin{aligned} & x \in A \cap (B \cup C) \text{ iff } (x \in A) \wedge [(x \in B) \vee (x \in C)] \\ & \text{iff } [(x \in A) \wedge (x \in B)] \vee [(x \in A) \wedge (x \in C)] \\ & \text{iff } x \in (A \cap B) \cup (A \cap C). \end{aligned}$$

Therefore $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ ■

2 Exercises

1. Prove that if $A \subseteq B, B \subseteq C, C \subseteq A$ then $A = B, B = C, C = A$.
2. Suppose m, n are natural numbers. Prove that $m\mathbb{Z} = n\mathbb{Z} \Leftrightarrow m = n$.
3. Prove that $S = T$ iff $\mathcal{P}(S) = \mathcal{P}(T)$.
4. Give succinct proofs, similar to the 2nd proof of Theorem 14:
   (a) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
   (b) $A \cap (B \cap C) = (A \cap B) \cap C$

Use a truth table to verify any tautologies you use in your proofs.