LECTURE 8 EXCERCISE SOLUTIONS

Problem. 1: Prove that Q is equivalent to $(P \implies Q) \wedge ((\sim P) \implies Q)$. Also, P is equivalent to $(\sim P) \implies Q \wedge (\sim Q)$.

Solution. I'm writing this out like this rather than giving truth tables to try to encourage you to think about how the parts of the statements relate to each other.

Equivalence 1: Q is equivalent to $(P \implies Q) \wedge ((\sim P) \implies Q)$
The conjunction is true if and only if both terms are true. Hence, $P \implies Q$ and $\sim P \implies Q$. Since either P or $\sim P$ is true, Q cannot be false (lest one of the implications be false). Therefore Q is true. Similarly, if Q is true, both implications (and therefore the conjunction) is true. Remember that both true and false truthfully imply true. If the conjunction is false, then one of the implications is false. The only way an implication can be false is if the right side is false, so Q must be false as well. Similarly, if Q is false, then one of the implications must be false, and the conjunction is false. Always verify both directions when proving equivalence.

Equivalence 2: P is equivalent to $(\sim P) \implies Q \wedge (\sim Q)$
Note that $Q \wedge (\sim Q)$ is always false, independent of what Q is. Hence if the implication is false, then $\sim P$ must be true, and therefor P is false. If the implication is true, then $\sim P$ is false, and P is true. You can similarly see what happens by plugging in P, for the full equivalence (verifying the inherent bidirectional).

Common Problems. No real problems here. Please show your work. I could write down the most incredible maths, but none of it means anything if I don't demonstrate it's true.

Problem. 2: Establish that $\{H_1, ..., H_n, P\} \vdash Q \rightsquigarrow \{H_1, ..., H_n, \sim Q\} \vdash (\sim P)$.

Solution. It's worth stating what this 'means'. Say you have a proof with hypotheses $H_1, H_2, ..., H_n, P$, that derives Q. Is it possible to write a proof starting with all the $H's$ and $\sim Q$, and deriving $\sim P$?

In any case, this is a fast application of principles from lecture 7.

By the deduction principle, $\{H_1, ..., H_n, P\} \vdash Q$ leads to $\{H_1, ..., H_n\} \vdash (P \implies Q)$. That is, given the hypotheses $H_1, ..., H_n$, we may prove $P \implies Q$. Noting that $P \implies Q$ is equivalent to $\sim Q \implies \sim P$, we can similarly derive from the hypotheses that $\sim Q \implies \sim P$.

Thus $\{H_1, ..., H_n\} \vdash (\sim Q \implies \sim P)$. Applying deduction again, in the opposite direction, we have that $\{H_1, ..., H_n, \sim Q\} \vdash (\sim P)$. Hence, chaining our steps together

$$\{H_1, ..., H_n, P\} \vdash Q \rightsquigarrow \{H_1, ..., H_n, \sim Q\} \vdash (\sim P)$$

Common Problems. I've no serious issues to bring up. As always, justify your steps.

One thing worth noting is that this is the second half of the contraposition principle, the other half of which is proven in the lecture notes. Since you're being asked to prove the contraposition principle, you can't really cite the contraposition principle, unless you are citing the direction already proven.

Problem. 3: Establish that $\{H_1, ..., H_n\} \vdash P \rightsquigarrow \{H_1, ..., H_n, \sim P\} \vdash (Q \wedge \sim Q)$.

Solution. This is even faster than the previous problem. By problem one, we have P is equivalent to $\sim P \implies (Q \wedge \sim Q)$. Hence, by equivalence, $\{H_1, ..., H_n\} \vdash P$ leads to $\{H_1, ..., H_n\} \vdash (\sim P \implies (Q \wedge \sim Q))$. Applying deduction in reverse, $\{H_1, ..., H_n, \sim P\} \vdash (Q \wedge \sim Q)$.

Common Problems. Again, not too much to complain about here. A number of people seemed to prove the problem backwards, starting on the right and ending on the left. If you look in the notes, you're essentially being asked to prove half the contradiction principle. If you prove this problem backwards, you're simply proving the direction given in the notes. Also again, you cannot cite the contradiction principle. People also lost points for citing principles and it not being clear how they applied, or what they were being applied to. I think it's worth saying that you should keep in mind what these mean in terms of proofs. If you can prove a statement from hypotheses, what else can you prove, and how?

Problem. 4: The biggest problem I had with this exercise was parsing it. For brevity's sake (and sanity), let $H = \{H_1, ..., H_n\}$.

Establish the following. $H \vdash Q \rightsquigarrow \{H, P\} \vdash Q, \{H, \sim P\} \vdash Q$.

Solution. Again, it's worth taking a moment to think about what this means. If you have a proof deriving Q from H, then you can construct a proof deriving Q from H and P, and you can construct one deriving Q from H and $\sim P$.

At a high level, a proof does remain valid if you add hypotheses. Every statement that depended on the original hypotheses will still be valid. The new hypotheses may lead to contradictions, but the steps by which you reached your original conclusion will remain valid. Hence, if you have a proof $H \vdash Q$, simply appending the hypothesis P gives a valid proof of $\{H, P\} \vdash Q$. Thus $\{H, P\} \vdash Q$. Similarly, $\{H, \sim P\} \vdash Q$.

At a less high level, note that starting with $H \vdash Q$, we can use the equivalence in problem 1 to state $H \vdash (P \implies Q) \wedge (\sim P \implies Q)$. Via conjunctive inference, we have that $H \vdash \{P \implies Q, \sim P \implies Q\}$. And we may separate that out (see lecture 6 exercises) to give $H \vdash P \implies Q$ and $H \vdash \sim P \implies Q$. Applying reverse deduction to each $\{H, P\} \vdash Q$ and $\{H, \sim P\} \vdash Q$, Hence,

$$H \vdash Q \rightsquigarrow \{H, P\} \vdash Q, \{H, \sim P\} \vdash Q$$

Question for you to consider - why can I just replace $\{H_1, ..., H_n\}$ with H like that?

Common Problems. There were a couple of recurring issues on this problem. I think many people had trouble parsing the problem, and stumbled as a result of that. Another group of people seem to have solved the problem backwards, going from right to left. Start with the hypothesis that $H \vdash Q$ and work from there. Another big problm people had was in not justifying their steps. Sometimes it's relatively trivial, like applying deduction in the previous two problems. Sometimes, less so. For instance, in order to apply something like conjunctive inference, you need a conjunction to work with - and a conjunction in your conclusion, not in your hypotheses. If you have some statement or proposition that isn't a conjunction, you need to justify why you can use conjunctive inference before you use it. Always try to be clear.