LECTURE 6 EXCERCISE SOLUTIONS

Problem. 1: Prove that $(R_1 \wedge R_2) \wedge R_3$ and $R_1 \wedge (R_2 \wedge R_3)$ are equivalent using a truth table. Give a convincing argument that all possible groupings of $R_1 \wedge ... \wedge R_n$ are equivalent.

Solution. The first part of this problem is just like a problem on the previous homework. In short, note that if any one of R_1, R_2, R_3 are false, both $(R_1 \wedge R_2) \wedge R_3$ and $R_1 \wedge (R_2 \wedge R_3)$. And if all are true, both are true. This extends to $R_1 \wedge ... \wedge R_n$. If any one of them is false, the smallest group that contains it is false. Then the smallest group that contains that group is false. And it boils all the way to the top. If they are all true, then the smallest grouping will be true, the grouping that contains it will be true, etc, extending up to the top. Note, this is only a convincing argument. Not a proof.

Common Problems. I took of a point from a lot of people, so I should explain myself. Many arguments went like this.

If any one of $R_1, R_2, R_3, ..., R_n$ is false, $R_1 \wedge ... \wedge R_n$ will be false. The only way for it to be true is for each one to be true

The problem with this is that, as it says in the lecture notes, $R_1 \wedge ... \wedge R_n$ is ambiguous. What does it mean? Does $R_1 \wedge R_2 \wedge R_3$ mean $R_1 \wedge (R_2 \wedge R_3)$ or does it mean $(R_1 \wedge R_2) \wedge R_3$? As you showed in your truth table, it doesn't matter - the two are equivalent. But before you can say anything about $R_1 \wedge ... \wedge R_n$, you have to decide what it means. And to decide that 'if any one is false, the whole thing will be false' is to implicitly assume beforehand that all groupings are equivalent. Which is what you are trying to justify. So to make a convincing argument, you have to justify why grouping doesn't matter.

Problem. 2: Give high level arguments for the following.

(a) $(H \vdash Q_1), (H \vdash Q_2), ..., (H \vdash Q_m) \rightsquigarrow H \vdash \{Q_1, ..., Q_m\}$

(b) $H \vdash Q \rightsquigarrow H \vdash Q_i, \; for \; i = 1, ..., m$

Solution. I wrote up this solution as I imagined the professor might, but I was fairly generous with the level of rigor I accepted.

(a) The citation principle says that, as applied to this problem, if we have proved that $H \vdash Q_i$, in any subsequent proof in which H occurs, we may write down Q_i. Starting out, we have $H \vdash Q_1$. After that, we may write down Q_1 in any proof in which H occurs.

Hence, in the proof that $H \vdash Q_2$, we may write down Q_1 (or in effect, concatenate the proofs), to yield $H \vdash \{Q_1, Q_2\}$. In this way, stepping down the list of stated hypotheses, we can build up to $H \vdash \{Q_1, ..., Q_m\}$.

(b) We are given that $H \vdash \{Q_1, ..., Q_m\}$. Given the proof of $\{Q_1, ..., Q_m\}$ from H, for any i, we may simply omit the statements/proofs of $Q_1, Q_2, ...Q_{i-1}, Q_{i+1}, ..., Q_m$, so as to leave a proof of $H \vdash Q_i$.

As an alternate approach, consider the statement $\{Q_1, ..., Q_m\} \vdash Q_i$. This is certainly true, since Q_i is contained in the hypotheses. Hence, using the citation principle,

$$(H \vdash \{Q_1, ..., Q_m\}), (\{Q_1, ..., Q_m\} \vdash Q_i) \rightsquigarrow H \vdash Q_i$$

Common Problems. As I'm grading this, I'm seeing a few persistent issues. For instance, people citing the citation principle, without stating how they are applying it to conclude what they want to conclude. Always show your logic. Why what you are doing does what you want it to do. Just because I claim to know or be able to do something doesn't mean I do or can. Another issue is that, in part (b), people go through some number of steps to conclude from $H \vdash \{Q_1, ..., Q_m\}$ that $H \vdash \{Q_1, ..., Q_m\}$. You don't need to go through any steps to conclude that, since you are given it. These people and others would generally then state $H \vdash \{Q_1, ..., Q_m\}$ leads to $H \vdash Q_i$ directly. That's the whole point of the problem, justifying that implication. Why can you pick out any one Q_i from the list?

Problem. 3: Give a high level argument that shows the citation principle can be iterated as follows:

$$H \vdash Q, Q \vdash R, R \vdash S \rightsquigarrow H \vdash S$$

Solution. Applying the citation principle in (3) once, we have that $H \vdash Q, Q \vdash R \rightsquigarrow H \vdash R$. Hence, our hypotheses can be substituted by $H \vdash R, R \vdash S$. Applying the citation principle again, we have $H \vdash S$. Therefore, $H \vdash Q, Q \vdash R, R \vdash S \rightsquigarrow H \vdash S$.

Common Problems. There were no serious problems here. If I took off points, I took off points because I couldn't understand what you were doing, you were making the problem more complicated than could be necessary, or you were simply doing something wrong.

Problem. 4: Give a high level proof that establishes:

$$(\{H_1, ..., H_n\} \vdash R_1 \wedge ... \wedge R_m) \rightsquigarrow (\{H_1, ..., H_n\} \vdash \{R_1, ..., R_m\})$$

Solution. To begin with, let $H = \{H_1, ..., H_n\}$. Note that we have the tautology, for any $i = 1, ..., m$, $R_1 \wedge ... \wedge R_m \implies R_i$. Hence, we have that $R_1 \wedge ... \wedge R_m \vdash R_i$. Then by the citation principle,

$$(H \vdash R_1 \wedge ... \wedge R_m), (R_1 \wedge ... \wedge R_m \vdash R_i) \rightsquigarrow (H \vdash R_i)$$

Noting that we have the above, for -each- R_i, we can use the result from problem 2,

$$(H \vdash R_1), ..., (H \vdash R_m) \rightsquigarrow H \vdash \{R_1, ..., R_m\}$$

An alternative proof of this might use modus ponens on the tautology, rather than utilizing it as a proof. I'm flexible.

Common Problems. I took off a lot of points on this one. To begin with, a lot of people were doing things that I simply did not understand, using hypotheses that seemed to come out of no where, or inferences that they did not justify. It seemed as though many people did not discover the necessary tautology. Another group of people appeared to simply prove the wrong thing. Answers were all over the map on this one.