LECTURE 3-4 EXCERCISE SOLUTIONS

Problem. 1: Prove $S^c = U \iff S = \{\}$ similarly to how $S = U \iff S^c = \{\}$ was proven.

Solution. Note that S will contain at least one element, unless $S^c = U$. This shows $S = \{\} \implies S^c = U$. Note too that if S^c contains -every- element, then no element can be in S. Hence, $S^c = U \implies S = \{\}$. Hence, $S^c = U \iff S = \{\}$.

Common Problems. I saw a lot of people only justifying one direction, but not the other. That's only half the problem! Another group of people proved the statement using $(S^c)^c = S$ which is true, but is not how you were asked to solve this problem. That's the next problem. Lastly, a lot of people used very vague language, where it wasn't clear what you were assuming, or what you were using to justify your conclusions. Always be clear. Another issue that popped up here and there was people giving a specific example of a universe U and a subset S, and proving the statement true for that example. You cannot prove a statement like this for all $S \subseteq U$ by example. You rarely can prove things by example, unless you're trying to disprove something.

Problem. 2: Given that $(S^c)^c = S$ and $S = U \iff S^c = \{\}$, deduce $S^c = U \iff S = \{\}$.

Solution. We know that $S = U \iff S^c = \{\}$ is true for any subset S of U. Given a subset S, S^c is also a subset of U, so it must be true that (and watch the parenthesis here), $(S^c) = U \iff (S^c)^c = \{\}$. Utilizing that $(S^c)^c = S$, we have that $S^c = U \iff S = \{\}$.

Common Problems. A lot of people looked at this problem and said things like, take $S = U \iff S^c = \{\}$ 'and compliment both sides'. What does that mean? How do you take the compliment of two sides of a proposition? You compliment sets, not propositions. At best, I would view complimenting both sides as taking $S^c = U^c \iff (S^c)^c = \{\}^c$. But that just reduces to $S^c = \{\} \iff S = U$. Nothing gained. What most of you meant seemed to be to take $S^c = U \iff (S^c)^c = \{\}$, which is right ... but very awkward. Whereas in the solution I've given, you're utilizing a true statement about sets on a single given set, here you are operating on a true statement. How do you know that once you operate on it, it will still be true? To assume that it will still be true is, in effect, to assume what you're trying to prove. And that, in mathematical terms, is bad. In a related issue, people would state what they want to prove, $S^c = U \iff S = \{\}$, substitute or in some way utilize $(S^c)^c = S$, and simplify it to $S = U \iff S^c = \{\}$ and conclude, since that is true, the original statement must be true. Logic doesn't work that way though. Do not assume what you want to prove. Also, again many people tried to prove by example. Not helpful here.

Problem. 3: Give a proof of $\sim (\forall x \in U)P(x) \iff (\exists x \in U)(\sim P(x))$ similar to the proof of $\sim (\exists x \in U)P(x) \iff (\forall x \in U)(\sim P(x))$.

Solution. Let S be the truth set for P in U. Noting that $S = U \iff S^c = \{\}$, we can state equivalently, $S \neq U \iff S^c \neq \{\}$. This is a true statement for any subset of U, but it has particular meaning if we take S to be the truthset of P. $S \neq U$ is equivalent to saying $\sim (\forall xinU)P(x)$, that is, it is not true that every x is in the truthset. $S^c \neq \{\}$ is equivalent to saying, the set of $x \in U$ for which $P(x)$ is not true is not empty. In other words, $(\exists x \in U) \sim P(x)$. Thus $S \neq U \iff S^c \neq \{\}$, true for any subset S of U, in the particular instance of S as the truthset of P, means $\sim (\forall x \in U)P(x) \iff (\exists x \in U)(\sim P(x))$ is true.

Common Problems. In this problem, there was a lot of vagueness. People would introduce the set S without ever saying what it meant in the problem. People would also seem to get lost in the quantifiers, and how they would relate to the truthset. If P is true for everything in U, then the truthset is all of U. If P isn't true for for everything in U, then the truthset isn't U, but it doesn't necessarily have to be $\{\}$. That is one possibilitiy, but it isn't the only one. The opposite of everything isn't nothing, it's simply not everything. Another problem people had was language, using $P(x)$ to refer to the truthset for P. sThis problem was a big favorite to be proved by example, but yet again, that approach doesn't cover the full generality of the statement you're trying to prove.

Problem. 4: Give an example of an open sentence $P(x)$ with universe U such that $(\exists x \in U)P(x)$ is false, but $\forall x \in U)P(x)$ is true.

Solution. Consider the example of $U = \{\}$, and P can be anything. Does there exist $x \in U$ so that $P(x)$ is true? No, because there are no $x \in \{\}$. And yet, for every $x \in U$, $P(x)$ is true - because there are no such x.

Common Problems. People either found this example or they didn't. I took off points if you claimed it was impossible, without trying to justify why it was impossible. I took off fewer points if you tried to justify why it was impossible. And I took off no points for the correct answer.