LECTURE 2 EXCERCISE SOLUTIONS

Note, the first two problems on this assignment are just proving things equivalent, using lots and lots of truth tables. If, in any problem, you didn't get that they are equivalent, then you should check your work. When I took of points on these two problems, it was usually for some computational misstep. Go over the truth tables for $\sim$, $\Longrightarrow$, $\vee$, $\wedge$. Know them by heart.

Problem. 3: The *inverse* of a formula f is the formula $\bar{\text{f}}$, obtained by negating all the variables. [For example, if f $= f(X_1, X_2, ..., X_n)$, $\bar{\text{f}}= f(\sim X_1, \sim X_2, ..., \sim X_n)$]. The *dual* $\tilde{\text{f}}$ of f is the negation of its inverse, thus $\tilde{\text{f}} = \sim \bar{\text{f}}$.

Show using De Morgan's laws that the dual $\tilde{\text{f}}$ is equivalent to the formula obtained from f by replacing all $\wedge$ with $\vee$ and vice versa.

Solution. If I had to guess, this is the problem most people are asking about. It is also the problem I took off the most points for. The issue was this: I read the problem as asking you to justify something in generality, for any formula f. What everyone did, across the board, was pick some example formula, and prove it for that example. So you get points for that, but you can't prove something like this by example. I wasn't looking for a precise proof, but I did want some kind of justification as to why it was universally true. So, here is how I thought about it.

Let the formula f be given by $f(X_1, X_2, ..., X_n) = g(X_1, X_2, ..., X_n) \wedge h(X_1, X_2, ..., X_n)$, where g and h are some formulas in the specified variables. Further, let's assume the statement is true for all formulas 'smaller' than f. Since f is composed of g and h, g and h are 'smaller' than f, and we assume the statement we're trying to prove is true on them. Let g' and h' be, respectively, the formulas obtained from g and h by exchanging $\vee$ with $\wedge$ and vice versa. Therefore, we are saying that $\tilde{\text{g}}$ = g', and $\tilde{\text{h}}$ = h'. In other words,

$$g'(X_1, X_2, ..., X_n) =\sim g(\sim X_1, \sim X_2, ..., \sim X_n)$$
$$h'(X_1, X_2, ..., X_n) =\sim h(\sim X_1, \sim X_2, ..., \sim X_n)$$

Then, in calculating $\tilde{\text{f}}$, we have that $\tilde{\text{f}} = \sim f(\sim X_1, \sim X_2, ..., \sim X_n)$. Substituting and using De Morgan's Laws, the following are all equivalent.

$$\sim f(\sim X_1, \sim X_2, ..., \sim X_n)$$
$$\sim (g(\sim X_1, ..., \sim X_n) \wedge h(\sim X_1, ..., \sim X_n))$$
$$(\sim g(\sim X_1, ..., \sim X_n)) \vee (\sim h(\sim X_1, ..., \sim X_n))$$
$$g'(X_1, ..., X_n) \vee h'(X_1, ..., X_n)$$

Note that in that last step, g' and h' swap all $\wedge$ and $\vee$, and we've swapped the last $\wedge$ in f with a $\vee$. Hence, -whatever- f looks like, in calculating the dual, we've effectively exchanged all $\wedge$ and $\vee$, and vice versa..

This is effectively an induction type-argument, with De Morgan's laws acting as a base case. Note, you'd also have to consider the case where $f = g \vee h$, but it works out to the same kind of thing.

Hence, in some generality, the dual of f is obtained by exchanging all $\vee$ with $\wedge$, and vice versa.

Common Problems. Note that in this problem, I was not looking for a specific proof, or an especially precise proof. What I was looking for was some kind of statement why this would be true in generality. For instance, note that the above proof is effectively just applying De Morgan's laws on -formulas-, rather than just variables. Justifying De Morgan's laws by themselves was insufficient, without explaining how that might generalize.

Problem. 4: Prove that if f is equivalent to g, then a) $\bar{f}$ is equivalent to $\bar{g}$, b) $\sim$f is equivalent to $\sim$g, c) $\tilde{f}$ is equivalent to $\tilde{g}$.

Solution. a) If f is given by $f(X_1, X_2, ..., X_n)$. Then $\bar{f}$ is given by $f(\sim X_1, ..., \sim X_n)$. Thinking in terms of a truth table, what this effectively does is to flip the f column upside down, so that (T, T, ..., T) now has $f(F, F, ..., F)$, and so on down the column. This turns the column for f upside down to yield the column for $\bar{f}$.

If f and g are equivalent, then they have equal columns in a truth table. Thus, when the columns are inverted, they stay equivalent. Thus $\bar{f}$ is equivalent to $\bar{g}$.

b) If f is equivalent to g, then for any assignment of the variables, $f(X_1, ..., X_n) = g(X_1, ..., X_n)$. Then, $\sim f(X_1, ..., X_n) = \sim g(X_1, ..., X_n)$. Thus for all possible assignments of variables, $\sim$f = $\sim$g.

c) Recall that $\tilde{f} = \sim\bar{f}$. By part a), we have that if f and g are equivalent, $\bar{f}$ and $\bar{g}$ are equivalent. By part b), we have that since $\bar{f}$ and $\bar{g}$ are equivalent, $\sim\bar{f}$ and $\sim\bar{g}$ are equivalent. Thus $\tilde{f}$ and $\tilde{g}$ are equivalent.

Common Problems. The key here is in part a. The reason is that taking the inverse operates on the variables inside of f and g, rather than the values of f and g themselves. Hence it is insufficient to say, *Since they are equivalent, doing the same thing to both of them will leave them equal*, or something similar. f and g may depend on their variables in different ways (very different looking statements can turn out to be equivalent to each other). Hence you have to make some statement regarding the effect on those variables - and the hint was to encourage you to think about what the operation does to the truth tables.