LECTURE 18 EXCERCISE SOLUTIONS

Problem. 1: Prove that if $a, d \in \mathbb{N}$, and k is an integer such that $a = dk$, then $k \in \mathbb{N}$. (4 Points)

Solution. We have that k is an integer. Suppose that k is not in $\mathbb{N}$, so $k \leq 0$. Rather, $k = 0$, or $k < 0$. If $k = 0$, then $a = dk = d0 = 0$. $a \in \mathbb{N}$, so $a \geq 1$. Hence, this is a contradiction, and $k \neq 0$. If $k < 0$, note that the product of a positive number (d) and a negative number (k) will be negative. Therefore, if $a = dk$, then a is negative. But again, $a \in \mathbb{N}$, so a is not negative. Contradiction! Therefore, k is not less than 0. Since k is not 0, and is not less than 0, we have that $k > 0$, or that $k \in \mathbb{N}$.

Common Problems. The most common mistake was the following. To argue that $k = a/d$, and $a/d \in \mathbb{N}$, therefore $k \in \mathbb{N}$. The problem is that the statement '$a/d \in \mathbb{N}$' is exactly the statement you are trying to prove. **Why** is the quotient of one natural number and another natural number still a natural number? Following the proof I give here, it's because the product of a non-natural number and a natural number cannot be a natural number. Another issue with this is that we're basically building towards the idea of division. We have a concept of 'divides', but we don't really yet have 'division'. What does it mean to take a divided by b, especially if b doesn't divide a? But any sort of proof that a positive/positive = positive is going to look basically like what I lay out here.

Another thing I saw that I must mention is a number of people made an argument that centered on the statement $(a \geq 1)/(d \geq 1) \in \mathbb{N}$. It's effectively the same argument as before, dividing two positive numbers must give you a positive number. The problem is still justifying that. The added difficulty here is that you cannot 'divide' propositions or statements. What would it mean to take $(1 < 2)/(4 = 4)$? I recognize what you were trying to say, but it is important to express your ideas in the right notation - and if you don't have the notation to express it, use words. I'm a big fan of words.

A last group of people used the claim that if d divides a, then $d \leq a$. This is true, and fine - but it is Lemma 5, which you are asked to prove in problem 2. I'm always troubled by using results before you've technically proven them. Some people solved this by proving problem 2, then problem 1, and that's fine. But in all cases, you really should make sure you're justified in using whatever claims of lemmas you use.

Problem. 2: Let a, b, d be natural numbers, and let $min(a, b)$ denote the smaller of a and b. Prove that if d is a divisor of a, then $d \leq a$. Also, if d is a common divisor of (a, b), then $d \leq min(a, b)$.

Solution. Suppose that d divides a. Then there exists a natural number k (natural, by the previous problem), so that $a = dk$. Noting that $1 \leq k$, and d is a natural number, therefore positive, we have that $d \leq dk$, or $d \leq a$.

Suppose that d is a divisor of the common divisor of a and b. This implies that d divides a, and d divides b, by the definition of common divisors, transitivity of divisibility, whatever you like. Using the claim just proven, we therefore have that $d \leq a$ and $d \leq b$. Noting that $min(a, b) = a$ or $min(a, b) = b$, in either case we get that $d \leq min(a, b)$.

Common Problems. I've no serious issues to mention here. A lot of people, in the course of proving this problem, reproved problem 1. You should note that from time to time, these problems you're asked to do are all connected in some way. Don't prove anything you don't need to, and don't reinvent the wheel.

Problem. 3: If $n \in \mathbb{Z}$, and $d \in \mathbb{N}$, then there are unique integers q, r such that $n = qd + r$ and $0 \leq r < d$. (4 Points)

Solution. There are two parts to this problem. One, finding a pair of integers q, r that work, and then proving that those are the only integers that work.

First, note that Theorem 8, as per the hint, gives a unique integer q such that $d(q+1) > n \geq dq$. Take that q. Then, note that equivalently, $dq+d > n \geq dq$, or rather that $d > n-dq \geq 0$. Defining $r = n - dq$, you have a pair of integers r, q such that $dq + r = n$, and $d > r \geq 0$.

Next, uniqueness. Suppose that $dg + s = n$, with $d > s \geq 0$. Without loss of generality, we may assume that $g \geq q$. Note that $dg + s = dq + r$, or $d(g - q) = (r - s)$. If $r = s$, we get that $g = q$ and we're done, so assume that $r \neq s$. Since we assumed that $g \geq q$, both sides of the equation must be non-negative, and indeed non-zero since $r \neq s$. Therefore, both sides of the equation are positive. Therefore, we have that d is a natural number that divides $r - s$, which is another natural number. However, by problem 2, that gives us that $d \leq r - s$ Note that $r - s \leq r$, since s is non-negative. Therefore, we have that $d \leq r - s \leq r < d$, therefore, $d < d$. This is a contradiction. Hence, it cannot be the case that $r \neq s$. Therefore, $r = s$, and as stated, $g = q$. Therefore, any such pair r, q is unique.

Common Problems. Points were mostly lost here for not justifying uniqueness, or not clearly stating some aspect of the problem - defining what r is, defining the bounds of things, asserting that there was a particular number in a certain bound, when in fact there were many. This wasn't an easy problem, but most people got the main points. The main obstacle was in not clearly expressing what it was you were doing.