LECTURE 17 EXCERCISE SOLUTIONS

Problem. 1: Prove the following: Suppose $P(n)$ is an open sentence with universe $\mathbb{Z}_{\geq l}$ such that

1. $P(l)$ is true *2.* $\forall k \geq l$, we have $P(k) \implies P(k+1)$.

Then $P(n)$ is true for all integers $n \geq l$.
(4 Points)

Solution. Let $m = l - 1$. Define an open sentence Q with universe $\mathbb{N}$, such that $Q(n) = P(n+m)$. Note then, that if we can prove that $Q(n)$ is true for all $n \in \mathbb{N}$, then we've proven that $P(n+m)$ is true for all $n \geq 1$, or all $n + m \geq 1 + m = l$ - in other words, letting $k = n + m$, $P(k)$ is true for all $k \geq l$. **If** we can prove $Q(n)$ is true for all $\mathbb{N}$.

Note that $Q(1) = P(1 + m) = P(1 + l - 1) = P(l)$ is true.

Note too, that if we assume $Q(k)$ is true, then $P(k + m)$ is true. Then, by assumption, $P(k + m + 1)$ is true. Therefore, $Q(k + 1)$ is true. Therefore we conclude, for all $k \in \mathbb{N}$, $Q(k) \implies Q(k + 1)$.

Hence, by the PMI, $Q(n)$ is true for all $n \in \mathbb{N}$. Hence, as discussed, $P(n)$ is true for all $n \geq l$.

Common Problems. There weren't too many problems here. For the most part, points were lost for not being clear about what you were doing, or not justifying what you were doing. People would write that $Q(k) = P(m + k)$ and $Q(k + 1) = P(m + k + 1)$, and end there, not concluding anything. Other people might argue that $P(l)$ was true, $P(l + 1)$ is true, $P(l + 2)$ is true, 'etc, and by PMI, $P(n)$ is true for all $n \geq l$. Remember, to apply the principle of mathematical induction, the key step is to show that the truth of the k-th step implies the truth of the (k+1)-th step. In the first mistake I list here, the equalities are true, but by stating them, you haven't verified the necessary implication. You need to justify the fact that $Q(k)$ implies $Q(k + 1)$, and you need to do it by citing the assumptions of the problem.

In the second mistake I list here, there are two problems. One, each of the statements listed is true, yes. But that isn't the important thing - only the fact that the truth of the k-th step implies the truth of the (k+1)-th step matters. But the more serious problem is that the principle of mathematical induction cannot be implied unless your basecase is at 1, not l. That's the whole point of this problem, generalizing the PMI in a nice way. I know that it looks trivial, but you need to build up your set of tools from things you know to be true. You know that the PMI is true, and so you use that to build up to the Generalized PMI, which you can't just state to be true.

Problem. 2: Prove the following: Suppose $P(n)$ is an open sentence with universe $\mathbb{Z}_{\geq l}$ such that

1. $P(l)$ is true *2.* $\forall k \geq l$, we have $P(l) \wedge P(l+1) \wedge ... \wedge P(k) \implies P(k+1)$.

Then $P(n)$ is true for all integers $n \geq l$.
(4 Points)

Solution. Define an open sentence R with universe $\mathbb{Z}_{\geq l}$ such that $R(n) = P(l) \wedge P(l+1) \wedge ... \wedge P(n)$.

Note that $R(l) = P(l)$, thus by assumption is true.

Note too, that $R(n+1) = P(l) \wedge P(l+1) \wedge ... \wedge P(n) \wedge P(n+1) = R(n) \wedge P(n+1)$

Note three, that by the stated assuptions, we have that $R(n) \implies P(n+1)$, using the definition we have for $R(n)$.

Assume $R(n)$ is true. Since $R(n) \implies P(n+1)$, $P(n+1)$ is true. Therefore, $R(n) \wedge P(n+1)$ is true. Therefore, $R(n+1)$ is true. Therefore, we have established that $R(n) \implies R(n+1)$, for all $n \geq l$.

Hence, we have that $R(l)$ is true, and for all $n \geq l$, that $R(n) \implies R(n+1)$. Using generalized induction as we proved in problem one, $R(n)$ is true for all $n \geq l$. Therefore, since $P(l) \wedge P(l+1) \wedge ... \wedge P(n)$ is true for all $n \geq l$, we may conclude that $P(n)$ is true, for all $n \geq l$. This last conclusion requires some citation of some conjunction result, or just noting that if any one of the $P(n)$ were false, the corresponding $R(n)$ would have to be false. Hence, $R(n) \implies P(n)$.

Common Problems. The grading comments on the previous problem apply here as well. The two things that I most took off for on this problem were ... well, the first one was rather curious. People would argue that $R(l) = P(l)$, and is therefore true. Then they would argue that $R(l+1)$ has to be true, and then stop and conclude. Yes, induction requires showing that if one step is true, then the next step is also true, but the strength in induction is that you consider a -general- step. In this case, because l is fixed, you can't just do l and $l+1$, because they are only two numbers, and you need to prove the statement for all numbers greater than or equal to l.

One important aspect of this problem is concluding about P once you have the result on R. It is not obvious nor trivial that if R is always true, then P is always true. You need to justify in some way that $R(n) \implies P(n)$ to complete this proof.

Problem. 3: Use PMI to prove that, for all $n \in \mathbb{N}$,

$$\frac{1}{2!} + \frac{2}{3!} + ... + \frac{n}{(n+1)!} = 1 - \frac{1}{(n+1)!}$$

(4 Points)

Solution. Base case, $n = 1$: $\frac{1}{2!} = \frac{1}{2} = 1 - \frac{1}{2} = 1 - \frac{1}{2!}$, hence the proposition is true for $n = 1$.

Assume the proposition is true for all numbers up to and including some n. Hence, we are assuming that for this specific n,

$$\frac{1}{2!} + \frac{2}{3!} + ... + \frac{n}{(n+1)!} = 1 - \frac{1}{(n+1)!}$$

is true.
The rest is mainly computation. Considering the LHS of the proposition for $n + 1$,

$$\frac{1}{2!} + \frac{2}{3!} + ... + \frac{n}{(n+1)!} + \frac{n+1}{(n+2)!}$$
$$=$$
$$1 - \frac{1}{(n+1)!} + \frac{n+1}{(n+2)!}$$
$$=$$
$$1 - \frac{n+2}{(n+2)!} + \frac{n+1}{(n+2)!}$$
$$=$$
$$1 - \frac{1}{(n+2)!}$$

Hence, if the proposition is true for n, the proposition is true for $n + 1$. Therefore, by the PMI, the relation holds for all $n \in \mathbb{N}$.

Common Problems. There's a very curious confusion of notation I see in these sort of induction problems. People like to define some function $P(n)$, which they then want to prove true for all n. What isn't clear, especially from the way I see $P(n)$ used, is whether $P(n)$ is the *statement* $\frac{1}{2!} + \frac{2}{3!} + ... + \frac{n}{(n+1)!} = 1 - \frac{1}{(n+1)!}$, or if it is one of the *expressions*, like $P(n) = \frac{1}{2!} + \frac{2}{3!} + ... + \frac{n}{(n+1)!}$. You cannot do arithmetic on statements. You can't add them together, you can't subtract things, etc. You can do that to expressions, however.

The other thing that bothers me a great deal is seeing the following - when performing the induction step, I see people say, 'assume true for n', then we want to prove:

$\frac{1}{2!} + \frac{2}{3!} + ... + \frac{n+1}{(n+2)!} = 1 - \frac{1}{(n+2)!}$. They then take that equation, and do something like the following:

$$\frac{1}{2!} + \frac{2}{3!} + ... + \frac{n}{(n+1)!} + \frac{n+1}{(n+2)!} = 1 - \frac{1}{(n+2)!}$$

$$(\frac{1}{2!} + \frac{2}{3!} + ... + \frac{n}{(n+1)!}) + \frac{n+1}{(n+2)!} = 1 - \frac{1}{(n+2)!}$$

$$1 - \frac{1}{(n+1)!} + \frac{n+1}{(n+2)!} = 1 - \frac{1}{(n+2)!}$$

$$1 - \frac{1}{(n+1)!} + \frac{n+1}{(n+2)!} + \frac{1}{(n+2)!} = 1$$

$$1 - \frac{1}{(n+1)!} + \frac{n+2}{(n+2)!} = 1$$

$$1 - \frac{1}{(n+1)!} + \frac{1}{(n+1)!} = 1$$

$$1 = 1$$

And then you conclude, since this is true, the original equation must be true as well. This is the wrong way to do this. What you've done, writing up the proof in this way, is to say that we want to prove B (the case for $n+1$). We show that $B \implies C$, C being that $1 = 1$. Therefore, since C is true, and $B \implies C$, B is true. And that makes no sense. Just because $B \implies C$ and C is true, you have no information about whether or not B is true. Therefore, you cannot conclude that the $n+1$ case is true.

One way to salvage this kind of thing is to include $\iff$ statements at every step of the way. You need to make it clear that each step is equivalent to the one previous, and thus what you are trying to prove is equivalent to proving $1 = 1$. One place that this sort of thing is very important is when using inequalities and bounding quantities, when two lines in a proof like this won't necessarily be equivalent. A better way though, and this is a general sort of thing you should keep in mind, is to only start with what you *know*. You know that the n case is true, so instead of starting by writing down the $n+1$ case, write down the n case, and work from that to derive the $n+1$ case. In similar terms to what I just argued, this takes the form of showing $A \implies B$ and A is true, therefore you know B is true. This is logically sound, the previous approach is not.

Problem. 4: Use the PGI to prove, for all $n \geq 2$, that

$$(\frac{2^2-1}{2^2})(\frac{3^2-1}{3^2})...(\frac{n^2-1}{n^2}) = \frac{n+1}{2n}$$

(4 Points)

Solution. Base case, $n = 2$: $\frac{2^2-1}{2^2} = \frac{4-1}{4} = \frac{3}{4} = \frac{2+1}{2*2}$, hence the proposition holds for $n = 2$.

Assume the proposition holds for all numbers up to and including some n. Hence, we are assuming for that particular n, that

$$(\frac{2^2-1}{2^2})(\frac{3^2-1}{3^2})...(\frac{n^2-1}{n^2}) = \frac{n+1}{2n}$$

is true.
Again, the rest is mainly computation. Consider the LHS of the above relation for $n + 1$,

$$(\frac{2^2-1}{2^2})(\frac{3^2-1}{3^2})...(\frac{n^2-1}{n^2})(\frac{(n+1)^2-1}{(n+1)^2})$$
$$=$$
$$(\frac{n+1}{2n})(\frac{(n+1)^2-1}{(n+1)^2})$$
$$=$$
$$(\frac{1}{2n})(\frac{(n+1)^2-1}{(n+1)})$$
$$=$$
$$\frac{n^2+2n}{(2n)(n+1)}$$
$$=$$
$$\frac{n(n+2)}{(2n)(n+1)}$$
$$=$$
$$\frac{n+2}{2(n+1)}$$

Hence, if the proposition holds true for n, it holds true for $n+1$. Therefore, by the PGI, the relation holds for all $n \geq 2$

Common Problems. Previous comments apply. Though I will add that in this problem, people were especially good at obfuscating what they were doing using lots of messy algebra and symbols going every which way.