LECTURE 16 EXCERCISE SOLUTIONS

Problem. 1: Prove that $1 + 2 + 3 + ... + n = n(n + 1)/2$. (4 Points)

Solution. To verify the base case, for $n = 1$, it suffices to show that 1 is equivalent to $1(1 + 1)/2$. Note that $1(1 + 1)/2 = 1 * 2/2 = 1$, so the base case is true.

Assume that, for some n, $1 + 2 + 3 + ... + n = n(n + 1)/2$ is true.

Consider the summation for $n + 1$. Note that $1 + 2 + 3 + ... + n + (n + 1) = (1 + 2 + 3 + ... + n) + (n + 1)$.

By the inductive hypothesis, $1 + 2 + 3 + ... + n + (n + 1) = n(n + 1)/2 + (n + 1)$

Note then, that $n(n+1)/2+(n+1) = (n/2+1)(n+1) = (n+2)(n+1)/2 = (n+1)((n+1)+1)/2$.

Therefore, $1 + 2 + 3 + ... + n + (n + 1) = (n + 1)((n + 1) + 1)/2$.

Therefore, since $1 + 2 + 3 + ... + n = n(n + 1)/2$ is true for $n = 1$, and since if it is true for n, it is true for $n + 1$, it is true for all natural numbers n.

Common Problems. There were no serious systematic errors here. If I took off points, I took them off because you were being unclear in what you were assuming, stating or assuming things in a way you shouldn't have, or were just being generally very hard to follow. Another possible source of mistakes was the algebra, but I think that was generally all right.

Problem. 2: Prove that $1^2 + 2^2 + 3^2 + ... + n^2 = n(n + 1)(2n + 1)/6$. (4 Points)

Solution. To verify the base case, for $n = 1$, it suffices to show that 1 is equivalent to $1(1 + 1)(2 * 1 + 1)/6$. Note that $1(1 + 1)(2 * 1 + 1)/6 = 1(2)(3)/6 = 1$, so the base case is true.

Assume that, for some n, $1^2 + 2^2 + 3^2 + ... + n^2 = n(n + 1)(2n + 1)/6$ is true.

Consider the summation for $n + 1$. Note that $1^2 + 2^2 + 3^2 + ... + n^2 + (n + 1)^2 = (1^2 + 2^2 + 3^2 + ... + n^2) + (n + 1)^2$.

By the inductive hypothesis, $1^2 + 2^2 + 3^2 + ... + n^2 + (n+1)^2 = n(n+1)(2n+1)/6 + (n+1)^2$

A bit of algebra follows.

$$n(n+1)(2n+1)/6+(n+1)^2$$
$$(n(2n+1)/6+(n+1))(n+1)$$
$$(2n^2+n+6n+6)(n+1)/6$$
$$(2n^2+7n+6)(n+1)/6$$
$$(2n+3)(n+2)(n+1)/6$$
$$(n+1)((n+1)+1)(2(n+1)+1)/6$$

Therefore, $1^2+2^2+3^2+...+n^2+(n+1)^2=(n+1)((n+1)+1)(2(n+1)+1)/6$.

Therefore, since $1^2+2^2+3^2+...+n^2=n(n+1)(2n+1)/6$ is true for $n=1$, and since if it is true for n, it is true for $n+1$, it is true for all natural numbers n.

Common Problems. Previous remarks apply. Especially with regards to the algebra.

Problem. 3: Prove that $1^3+2^3+3^3+...+n^3=(n(n+1)/2)^2$. (4 Points)

Solution. To verify the base case, for $n=1$, it suffices to show that 1 is equivalent to $(1(1+1)/2)^2$. Note that $(1(1+1)/2)^2=(1(2)/2)^2=1^2=1$, so the base case is true.

Assume that, for some n, $1^3+2^3+3^3+...+n^3=(n(n+1)/2)^2$ is true.

Consider the summation for $n+1$. Note that $1^3+2^3+3^3+...+n^3+(n+1)^3=(1^3+2^3+3^3+...+n^3)+(n+1)^3$.

By the inductive hypothesis, $1^3+2^3+3^3+...+n^3+(n+1)^3=(n(n+1)/2)^2+(n+1)^3$

A bit of algebra follows.

$$(n(n+1)/2)^2+(n+1)^3$$
$$((n/2)^2+(n+1))(n+1)^2$$
$$(n^2/4+n+1)(n+1)^2$$
$$(n^2+4n+4)(n+1)^2/4$$
$$(n+2)^2(n+1)^2/4$$
$$((n+2)(n+1)/2)^2$$
$$((n+1)((n+1)+1)/2)^2$$

Therefore, $1^3+2^3+3^3+...+n^3+(n+1)^3=((n+1)((n+1)+1)/2)^2$.

Therefore, since $1^3 + 2^3 + 3^3 + ... + n^3 = (n(n+1)/2)^2$ is true for $n = 1$, and since if it is true for n, it is true for $n+1$, it is true for all natural numbers n.

Common Problems. Previous remarks apply.

Problem. 4: Prove that 8 divides $9^n - 1$ for all natural numbers n. (4 Points)

Solution. Consider the base case, for $n = 1$. In that case, $9^n - 1 = 9^1 - 1 = 9 - 1 = 8$, and 8 certainly divides 8.

Assume that, for some n, 8 divides $9^n - 1$. That is, there is some number k such that $8 * k = 9^n - 1$.

Consider $9^{n+1} - 1$. Note that $9^{n+1} - 1 = 9 * 9^n - 1$. By assumption, we know that $8 * k = 9^n - 1$. Therefore, $9^n = 8 * k + 1$. Therefore, $9^{n+1} - 1 = 9 * (8 * k + 1) - 1 = 9 * 8 * k + 9 - 1 = 9 * 8 * k + 8 = 8 * (9 * k + 1)$. Therefore, there exists a number h, namely $h = 9 * k + 1$, such that $8 * h = 9^{n+1} - 1$. Therefore, 8 divides $9^{n+1} - 1$.

Since 8 divides $9^n - 1$ for $n = 1$, and if it holds for n, it holds for $n+1$, we have therefore that 8 divides $9^n - 1$ for all natural numbers n.

Common Problems. There was one very peculiar solution people kept giving, involving taking $9 * 9^n - 1$ and factoring it as $(9 - 1) * (9^n - 1)$. This is very strange, and not at all right, since if you set those equal to each other, you get that $9^n = -7$, which makes less than sense. Other errors involved, for the most part, confusion about the nature of division and what it means 'to divide'.