LECTURE 12 EXCERCISE SOLUTIONS

Some words before I begin. A lot of what you were asked to prove in this assignment is really pretty straightforward. We all 'know' that when you have an inequality, multiplying each side by a negative flips the direction of the inequality. The point of this lecture and this assignment is to really pin down why that is true. What it means to 'flip the inequality', or whatever rule you like. The point of any problem is to take the axioms and definitions (what does $>$ really *mean*? How can you prove anything about inequalities before you know that?) and prove known results. Proofs arguing by what we 'know' to be true about how inequalities work are therefore usually wrong, since in many cases you are assuming the very thing you want to prove.

Problem. 1: Prove that the product of two negative numbers is positive. (4 points)

Solution. Let x, y be negative real numbers. Thus $-x, -y$ are positive real numbers. Therefore their product, $(-x) * (-y) = (-1)^2 * x * y = x * y$, is also positive. Therefore $x * y$ is positive. Therefore, the product of two negative numbers is positive.

An alternative approach is to simply consider the inverse of one of the numbers, x or y, and use the result on the product of a positive and a negative.

Common Problems. No serious issues here. A couple of people tried to argue some kind of proof by contraposition, but got caught up in the negations. The negation of two numbers being negative is that *at least one* of the numbers is positive. All the errors that arose seemed to be of that type, misinterpreting some logical step.

Problem. 2: Prove that if $x > y$ and $z < 0$, then $xz < yz$. (4 points).

Solution. We have that $x - y$ is positive, and that z is negative. Therefore $z * (x - y)$ is negative. Therefore $-z * (x - y) = z * (y - x)$ is positive. Therefore $yz - xz$ is positive. Therefore $yz > xz$.

Common Problems. One thing I saw a group of people doing was describing this problem in terms of geometric effects on the number line. Scaling 'distances' and 'flipping the line'. That's all well and good, but you really need to describe what these symbols and ideas mean rigorously and axiomatically. Geometric intuition is good, but you need to put it on a rigorous foundation. One other thing I saw a few people do is make some kind of statement to the effect of 'since $x > y$, x is positive and y is negative, and work out things from there.

Just from knowing that $x > y$, you have no added information about y other than how it relates to x. You don't know how it relates to 0, so you can't say whether it is positive or negative. $2 > 1, 1 > -1, -1 > -2$, all true, all with different signs for x and y. And I didn't even bring in 0.

Problem. 3: Prove the following. Two points each.
(a) $x \geq y$ iff $x - y \geq 0$
(b) If $x \geq y$ then $x + z \geq y + z$ for all real numbers x, y, z
(c) If $x \geq 0$ and $y \geq 0$ then $xy \geq 0$
(d) If $x \geq y$ and $z \geq 0$ then $xz \geq yz$

Solution. These all follow mostly from definitions and theorems proven - if you follow the definitions correctly.

(a) $x \geq y$ iff ($x > y$ or $x = y$) iff ($x - y > 0$ or $x - y = 0$) iff $x - y \geq 0$

(b) If $x \geq y$, then ($x > y$ or $x = y$), then ($x+z > y+z$ or $x+z = y+z$), then $x+z \geq y+z$. Note the use of the fact that given a strict inequality, I can add any number z to both sides. And naturally, it doesn't change the equality.

An alternative proof of (b) might use the following. If $x \geq y$, then by (a), we have that $x - y \geq 0$. Since $x - y = (x + z) - (y + z)$, we have that $(x + z) - (y + z) \geq 0$. Then again applying (a), we have that $x + z \geq y + z$.

(c) If $x \geq 0$ and $y \geq 0$, then ($x > 0$ or $x = 0$) and ($y > 0$ or $y = 0$). This presents four possible cases you have to consider for possible x, y values. However, you can reduce it to the following two cases. If $x > 0$ and $y > 0$, then x and y are both positive, and therefore $x * y$ is positive. Therefore $x * y > 0$. If x *or* y is 0 (thus encompassing the remaining three cases), then $x * y = 0$. Therefore, no matter the case, we have that $x * y > 0$ or $x * y = 0$. Therefore, no matter the case, if $x \geq 0$ and $y \geq 0$, then $x * y \geq 0$.

(d) If $x \geq y$ and $z \geq 0$, then ($x > y$ or $x = y$) and ($z > 0$ or $z = 0$). Note again, there are technically four cases to consider, and you can do them out individually to verify that in each case, $xz > yz$ or $xz = yz$.

Alternatively, consider the following. By (a), we have that $x - y \geq 0$. By (c), we have therefore that $z(x - y) \geq 0$. Therefore $xz - yz \geq 0$. Therefore, applying (a) again, $xz \geq yz$.

Common Problems. A big problem people had here was not finishing the proof. A biconditional like (a) has two implications, both of which need to be proven. One way to avoid this is to make every step of your proof a biconditional as I did here, in which case your proof reads backwards and forwards. But if you progress in one direction from assumptions to conclusion, you have only proved the biconditional in one direction. Similarly, depending on how you did it, any one of these proofs required a lot of cases. You need to consider every single case for a complete proof, or justify why what you are considering covers all possible cases.

One thing that really bothered me occurred in (b). Many people had a step in their proof that amounted to adding something to each side of a $\geq$-relation, ignoring the fact that this is what we're trying to prove - you can't use it in your proof! Never assume what you want to prove.

Another thing I noticed as several people using theorems and conclusions about $>$ relations on $\geq$ relations. This is incorrect - they are two fundamentally different things. Related, yes, but $\geq$ has a very specific definition which introduces the cases, which you must deal with.