LECTURE 11 EXCERCISE SOLUTIONS

Problem. 1: Prove that if x is rational and y is irrational, then $x + y$ is irrational.

Solution. Following the hint, we are being asked to prove that $(x\ rational) \wedge (y\ irrational) \implies (x + y\ irrational)$. This is equivalent to proving $(x\ rational) \wedge \sim (x + y\ irrational) \implies \sim (y\ irrational)$. Or, to put it in more natural terms, $(x\ rational) \wedge (x + y\ rational) \implies (y\ rational)$. This equivalent statement is very easy to prove.

Assume x is rational. Therefore, there exist some integers a, b such that $x = \frac{a}{b}$. Also assume that $x + y$ is rational. Therefore, there exist some integers c, d such that $x + y = \frac{c}{d}$. Note then that $y = (x + y) - x$, or equivalently,

$$y = \frac{c}{d} - \frac{a}{b} = \frac{bc - ad}{db}$$

Hence, we have exhibited integers e, f (namely $e = bc - ad$ and $f = db$) such that $y = \frac{e}{f}$. Hence, y is rational.

Therefore, given the equivalency of the statments in my first paragraph, we have that: if x is rational and y is irrational, then $x + y$ is irrational.

Problem. 2: Prove there are infinitely any irrational numbers.

Solution. I can think of two ways to do this off the top of my head. In either case, you start with the fact $\sqrt{2}$ is irrational.

Solution 1: Consider, for any natural number n, the number $n + \sqrt{2}$. We know that n is rational, and $\sqrt{2}$ is irrational. Therefore by the result in problem 1, $n + \sqrt{2}$ is irrational. So far, this gives you a means of constructing irrational numbers. It remains to show that, by this method, you construct infinitely many of them. Since there are infinitely many natural numbers, the only way this could fail to produce infinitely many irrational numbers is if the irrational numbers produced in this way were not distinct.

Suppose that for natural numbers n, m, that $n + \sqrt{2} = m + \sqrt{2}$. By 'subtraction', we get that $n = m$. What this tells us is that any natural number, by this method, produces a distinct irrational number from all the others. Hence, since there are infinitely many distinct natural numbers, this method produces infinitely many distinct irrational numbers.

Solution 2: Similarly, consider for any natural number n, the number $n\sqrt{2}$. In this case, you would need to justify that the product of a rational and an irrational is irrational (assume it is rational, divide by n to reach a contradiction). And then similarly, you need to make reference to the fact that this produces infinitely many distinct irrational numbers.

Common Problems. The sort of twiddly bit here is proving that this produces infinitely many distinct natural numbers. Any time you have to justify a step by something as basic as 'subtraction', it's worth asking whether you need to justify that step at all. Nevertheless, you need to make some reference to the fact that, however you went about this problem, infinitely many irrational numbers were produced.

Problem. 3: Prove that if k is a natural number, then $k, k+1$ have no common divisors other than 1.

Solution. The hint recommends proof by contradiction, but I would argue that it isn't strictly necessary. Very often, a proof by contradiction can easily turn into a direct proof by changing a couple of words. Anyway.

Suppose that d is a natural number that divides both k and $k+1$. Therefore, there are some natural numbers a, b such that $k = d * a$ and $k + 1 = d * b$. Consider that second equation, $k + 1 = d * b$. Substituting in $d * a$ for k, we are left with

$$d * a + 1 = d * b$$

Or

$$d * b - d * a = 1$$

The punchline here is that we get

$$d(b - a) = 1$$

Here we may state simply that 1 has no divisors other than itself and itself. Therefore, the above equation is telling us that $d = 1$.

Since, starting out, we let d be an arbitrary natural number that divides $k, k+1$, this shows that any natural number that divides $k, k+1$ must be equal to 1. Therefore, $k, k+1$ have no common divisors other than 1.

The proof by contradiction version of this would look almost exactly the same, but with the added assumption initially that $d > 1$, and concluding in a contradiction by showing that $d = 1$. Instead, I've proven directly that any divisor must be equal to 1. But it's all the same, ultimately.