

Solutions to review problems.

(1)

FIRST SET OF PROBLEMS

1

- (20) 1. Evaluate the indicated limits exactly. Give evidence to support your answers.

a) $\lim_{x \rightarrow 3} \frac{x^2 - x - 6}{x - 3}$

b) $\lim_{x \rightarrow -2^-} \frac{x^2 - x - 6}{|x + 2|}$

c) $\lim_{x \rightarrow \infty} \frac{4x - 7}{e^{3x}}$

d) $\lim_{x \rightarrow 0} \frac{\sin(3x)}{\cos(2x)}$

$$(a) = \lim_{x \rightarrow 3} \frac{(x-3)(x+2)}{(x-3)} = \lim_{x \rightarrow 3} x+2 = 5$$

(b) if $x < -2$, then $|x+2| = -(x+2)$ so the limit

$$\lim_{x \rightarrow -2^-} \frac{(x-3)(x+2)}{-(x+2)} = \lim_{x \rightarrow -2^-} -(x-3) = -(-5) = 5$$

(c) by L'Hopital's rule thus $= \lim_{x \rightarrow \infty} \frac{4}{3e^{3x}} = 0$

(d) $\lim_{x \rightarrow 0} \sin(3x) = 0$, $\lim_{x \rightarrow 0} \cos(2x) = 1$, so the limit of the quotient
is 0.

2. Use the definition of derivative to compute $f'(x)$ if
 $f(x) = \frac{1}{x^2}$. Thus $\lim_{\Delta x \rightarrow 0} \frac{\frac{1}{(x+\Delta x)^2} - \frac{1}{x^2}}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{\frac{x^2 - (x+\Delta x)^2}{x^2(x+\Delta x)^2}}{\Delta x}$
 $= \lim_{\Delta x \rightarrow 0} \frac{-2x\Delta x - (\Delta x)^2}{(\Delta x)(x^2)(x+\Delta x)^2} = \lim_{\Delta x \rightarrow 0} \frac{-2x + \Delta x}{x^2(x+\Delta x)^2} = \frac{-2x}{x^4} = -\frac{2}{x^3}$.

(2)

- (22) 3. Compute the derivatives with respect to x of the following functions.
Please do not simplify the answers.

a) $\cos(x^4 + 3)$
 b) $(e^{7x} + 3x^4) \left(\sqrt{x+5} + \frac{2}{x^3} \right)$
 c) $\frac{x^3 + 2}{5 \ln x}$
 d) $\int_{-3}^x \sin(t^3) dt$

(a) $-\sin(x^4 + 3)(4x^3)$ - chain rule

(b) $(7e^{7x} + 12x^3) \left(\sqrt{x+5} + \frac{2}{x^3} \right) + (e^{7x} + 3x^4) \left(\frac{1}{2\sqrt{x+5}} - \frac{6}{x^4} \right)$
- product rule

(c) $\frac{(3x^2)(5 \ln x) - (x^3 + 2)\left(\frac{5}{x}\right)}{(5 \ln x)^2}$ - quotient rule

(d) $\sin x^3$ - 2nd form of Fundamental Theorem of Calculus

- (8) 4. Suppose $f(x) = (x^2 + 9)^{200} - (17 - x^3)^{301}$.

- a) Compute $f'(x)$. Explain why the following statement is correct: if $x > 0$, then $f'(x) > 0$.
 b) Use calculus to explain why $f(76) > f(23)$. You must quote a specific result from this course and explain its relevance. Your answer to a) may be useful here.

(a) $f'(x) = 200(x^2 + 9)^{199}(2x) - (17 - x^3)^{300}(-3x^2)$
 $= 400x(x^2 + 9)^{199} + 3x^2(17 - x^3)^{300}$

If $x > 0$ both summands are > 0 and so $f'(x) > 0$

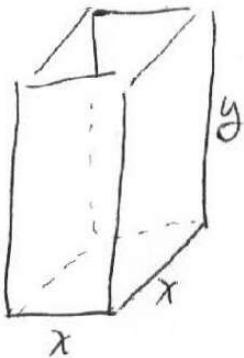
- (b) Since $f'(x) > 0$ for $x > 0$, $f(x)$ is increasing on $[0, \infty)$.

Since $76 > 23$ thus means $f(76) > f(23)$. This comes from the mean value theorem: $\frac{f(76) - f(23)}{76 - 23} = f'(c)$ for

some c in $(23, 76)$. Then $f'(c) > 0$ so

$$f(76) - f(23) = (53)f'(c) > 0.$$

- (16) 5. A rectangular box with a square base is to be made from two different materials. The material for the top and four sides costs \$1 per square foot, while the material for the bottom costs \$2 per square foot. If you can spend \$196 on materials, what dimensions will maximize the volume of the box?



$$V = x^2 y$$

Cost of bottom $2x^2$

Cost of each side xy

Cost of top x^2

Total cost $3x^2 + 4xy$

$$\text{So } 3x^2 + 4xy = 196$$

$$4xy = 196 - 3x^2$$

$$y = \frac{196 - 3x^2}{4x} = 49x^{-1} - \frac{3}{4}x$$

$$\text{Thus } V = x^2 y = 49x - \frac{3}{4}x^3$$

$$\frac{dV}{dx} = 49 - \frac{9}{4}x^2$$

$$\text{If } \frac{dV}{dx} = 0 \quad \text{then} \quad \frac{9}{4}x^2 = 49$$

$$x^2 = 49 \left(\frac{4}{9}\right)$$

$$x = 7\left(\frac{2}{3}\right) = \frac{14}{9}$$

$$y = \frac{91}{3}$$

- (10) 6. Find equations for all horizontal and vertical asymptotes of $f(x) = \frac{3+2e^{2x}}{5-7e^{2x}}$.

$$\lim_{x \rightarrow \infty} \frac{3+2e^{2x}}{5-7e^{2x}} = \frac{1}{4} \quad \lim_{x \rightarrow \infty} \frac{-14e^{2x}}{-14e^{2x}} = -\frac{2}{7} \quad \text{horizontal asymptotes:}$$

$$\lim_{x \rightarrow -\infty} \frac{3+2e^{2x}}{5-7e^{2x}} = \frac{3}{5}$$

$$y = -\frac{2}{7}$$

$$y = \frac{3}{5}$$

$$\text{If } 5-7e^{2x}=0, \quad e^{2x}=\frac{5}{7}, \quad x=\frac{1}{2}\ln\left(\frac{5}{7}\right) \leftarrow \text{vertical asymptote}$$

- (10) 7. a) Give an example of a function which is not continuous. Explain why your example is not continuous.
 b) Give an example of a function which is not differentiable. Explain why your example is not differentiable.

$$(a) f(x) = \begin{cases} 0 & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$

Then $\lim_{x \rightarrow 0} f(x) = 0 \neq f(0)$ so f is not continuous at 0.

(b) $f(x) = |x|$ is not differentiable at 0 since

$$\lim_{\Delta x \rightarrow 0^+} \frac{f(\cancel{\Delta x}) - f(0)}{\Delta x} = \lim_{\Delta x \rightarrow 0^+} \frac{|\Delta x|}{\Delta x} = \lim_{\Delta x \rightarrow 0^+} \frac{\Delta x}{\Delta x} = 1$$

$$\lim_{\Delta x \rightarrow 0^-} \frac{f(\Delta x) - f(0)}{\Delta x} = \lim_{\Delta x \rightarrow 0^-} \frac{|\Delta x|}{\Delta x} = \lim_{\Delta x \rightarrow 0^-} \frac{-\Delta x}{\Delta x} = -1.$$

2

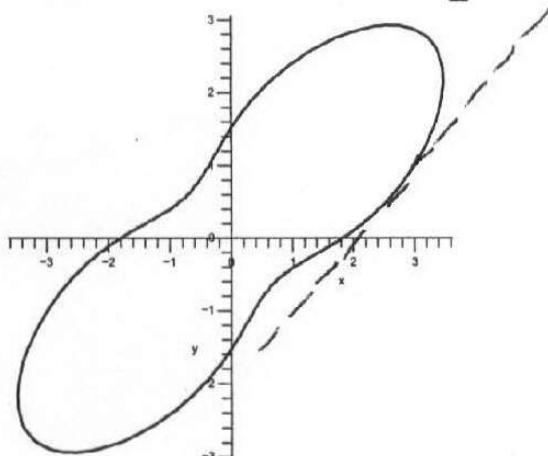
- (12) 8. Suppose that y is implicitly defined as a function of x by the equation $x^4 - 24xy + 2y^4 = 11$.

a) Find $\frac{dy}{dx}$ in terms of y and x .

- b) Find an equation for the line tangent to the graph at the point $P = (3, 1)$ which is on the graph.

Note There are some **Possibly useful numbers** on the formula sheet.

- c) The program Maple gives the image shown to the right when asked to graph the equation. Sketch the tangent line found in b) on the image.



$$(a) 4x^3 - 24y - 24x \frac{dy}{dx} + 8y^3 \frac{dy}{dx} = 0$$

$$\therefore \frac{dy}{dx} = \frac{24y - 4x^3}{8y^3 - 24x}$$

$$(b) \text{ At } (3, 1), \frac{dy}{dx} = \frac{21}{16} \text{ so the equation is } y - 1 = \frac{21}{16}(x - 3)$$

(c) See the dashed line on the diagram

(5)

- (18) 9. Suppose that $f'(x)$, the derivative of $f(x)$, is given by this formula:
 $f'(x) = (x+2)x^2(x-1)^3$.

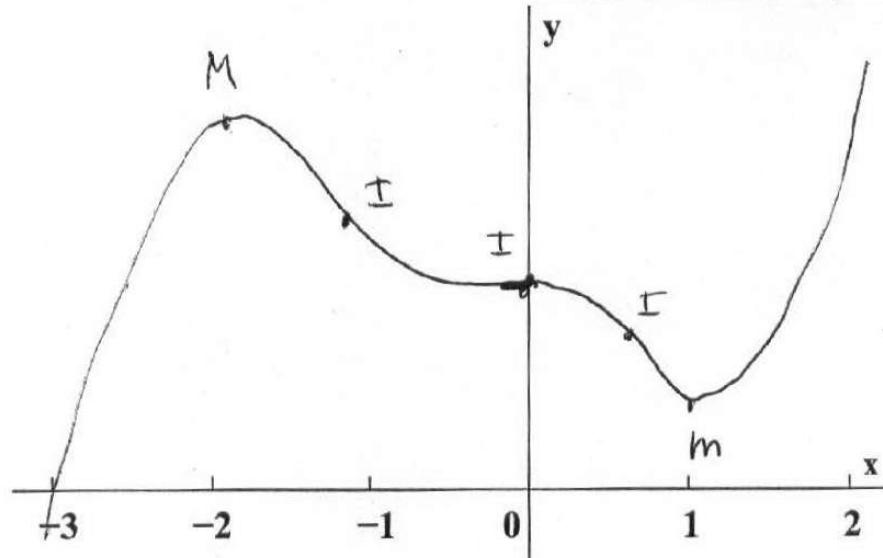
Note Read the formula carefully. Please do not try to find a formula for $f(x)$: this is not requested and will not help you answer any part of the problem.

a) What are the critical numbers of $f(x)$? For each critical number, explain why the associated critical point is a relative minimum, a relative maximum, or neither. Briefly support your answers using calculus.

b) Sketch a graph of $y = f(x)$ showing the features found in a) on the axes given. The graph should be as simple as possible. Label each relative maximum with M and label each relative minimum with m.

c) How many inflection points does your graph of $y = f(x)$ have? Label each inflection point with I on the graph drawn. Please do not compute $f''(x)$: use the graph.

Number of inflection points _____



(a) Critical numbers $-2, 0, 1$.

Note that $f'(x) > 0$ if $x < -2$

$f'(x) < 0$ if $-2 < x < 0$

$f'(x) < 0$ if $0 < x < 1$

$f'(x) > 0$ if $x > 1$

Then by the first derivative test the critical point associated to -2 is a relative maximum, the critical point associated to 1 is a relative minimum, and the critical point associated to 0 is neither.

(6)

- (12) 10. A spherical hot air balloon is being filled with air at the rate of 200 cubic feet per minute. At what rate is the radius of the balloon increasing when the balloon has 1000 cubic feet of air in it?

Let V be the volume, r the radius.

$$\text{Then } V = \frac{4}{3}\pi r^3 \text{ and so } \frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

We are given $\frac{dV}{dt} = 200 \text{ ft}^3/\text{min}$. Now when

$$V=1000 \text{ we have } 1000 = \frac{4}{3}\pi r^3 \text{ so } r^3 = \frac{750}{\pi},$$

$$r = \left(\frac{750}{\pi}\right)^{\frac{1}{3}}. \quad \text{Thus } 200 = \frac{dV}{dt} = 4\pi \left(\frac{750}{\pi}\right)^{\frac{2}{3}} \frac{dr}{dt},$$

$$\frac{dr}{dt} = \frac{200}{4\pi^{\frac{1}{3}}(750)^{\frac{2}{3}}} = 50\pi^{-\frac{1}{3}}(750)^{-\frac{2}{3}} \frac{dr}{dt}$$

- (12) 11. Suppose $f(x) = 3x^4 - 8x^3 - 18x^2 + 2$.

a) Find the exact maximum and minimum values of $f(x)$ for x in the interval $[-2, 2]$. Briefly explain your conclusions using calculus.

Note There are some **Possibly useful numbers** on the formula sheet.

b) If $[-2, 2]$ is the domain of the function $f(x)$, describe the range of $f(x)$ precisely. You may use your results in a) here. You must quote a specific result from this course and explain its relevance.

$$\begin{aligned} a) f'(x) &= 12x^3 - 24x^2 - 36x \\ &= 12x(x^2 - 2x - 3) = 12x(x-3)(x+1). \end{aligned}$$

Thus the critical numbers are $-1, 0, 3$. Only -1 and 0 are in the interval $[-2, 2]$. Thus an absolute maximum or minimum can occur only at one of $-2, -1, 0, 2$.

Now $f(-2) = 42$, $f(-1) = -5$, $f(0) = 2$, $f(2) = -86$. Thus the absolute minimum is -86 at $x=2$, the absolute maximum is 42 at $x=-2$.

(b) The range is $[-86, 42]$. This follows from the Intermediate Value Theorem.

(7)

#12 Skipped

(13) 13. Find the following indefinite integrals.

a) $\int \left(5\sqrt{x} - \frac{3}{x^2}\right) dx$

b) $\int e^{2x+x^2}(1+x) dx$

c) $\int (\cos(x+3) + \sin(2x)) dx$

$$\begin{aligned}
 (a) \quad \int 5x^{1/2} - 3x^{-2} dx &= 5\left(\frac{2}{3}\right)x^{3/2} - 3(-1)x^{-1} + C \\
 &= \frac{10}{3}x^{3/2} + 3x^{-1} + C
 \end{aligned}$$

(b) Let $u = 2x+x^2$ so $du = (2+2x)dx$.

Then the integral is $\int e^u \frac{du}{2} = \frac{1}{2}e^u + C = \frac{1}{2}e^{2x+x^2} + C$

$$\begin{aligned}
 (c) \quad * \text{ First consider } \int \sin(2x) dx \quad \text{If } u = 2x, du = 2dx \text{ so} \\
 \text{thus integral is } \frac{1}{2} \int \sin u du = -\frac{1}{2} \cos u + C = -\frac{1}{2} \cos(2x) + C.
 \end{aligned}$$

Then the given integral is $\sin(x+3) - \frac{1}{2} \cos(2x) + C$.

SECOND SET OF PROBLEMS

1 A

- (10) 1. Evaluate the indicated limits exactly. Give brief evidence supporting your answers which is not based on a calculator graph or calculator computations.

a) $\lim_{x \rightarrow \infty} \frac{2x^2 - 5}{3x^2 + 1}.$

b) $\lim_{x \rightarrow 4} \frac{x-4}{\sqrt{x}-2}.$

a) by L'Hopital's rule thus $\lim_{x \rightarrow \infty} \frac{4x}{6x} = \frac{2}{3}$

b) $\lim_{x \rightarrow 4} \left(\frac{x-4}{\sqrt{x}-2} \right) \left(\frac{\sqrt{x}+2}{\sqrt{x}+2} \right) = \lim_{x \rightarrow 4} \frac{(x-4)(\sqrt{x}+2)}{x-4} = \lim_{x \rightarrow 4} \sqrt{x}+2 = 4$

OR, by L'Hopital's rule

$$\lim_{x \rightarrow 4} \frac{1}{\frac{1}{2\sqrt{x}}} = \lim_{x \rightarrow 4} 2\sqrt{x} = 4$$

- (12) 2. Find the equations of all vertical and horizontal asymptotes of the function

$$f(x) = \frac{3e^x + 5}{7e^x - 2}.$$

Computations with exp and log should be simplified as much as possible. A numerical approximation like 1.40135 is **not** acceptable.

$$\lim_{x \rightarrow \infty} f(x) = \stackrel{L'H}{\lim_{x \rightarrow \infty}} \frac{3e^x}{7e^x} = \frac{3}{7} \quad \text{so} \quad y = \frac{3}{7} \text{ is a horizontal asymptote.}$$

$$\lim_{x \rightarrow -\infty} f(x) = \stackrel{L'H}{\lim_{x \rightarrow -\infty}} \frac{3e^x + 5}{7e^x - 2} = -\frac{5}{2} \quad \text{so} \quad y = -\frac{5}{2} \text{ is a horizontal asymptote.}$$

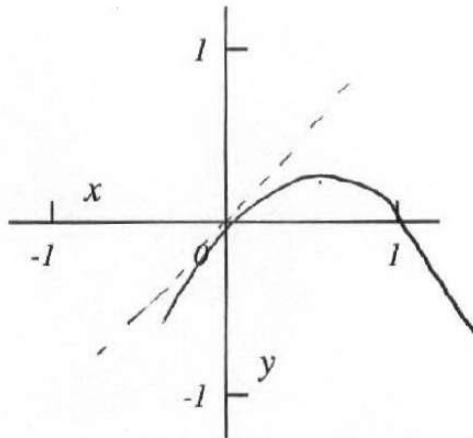
If $7e^x - 2 = 0$ then $7e^x = 2$ so $e^x = \frac{2}{7}$, $x = \ln(\frac{2}{7})$. Thus $x = \ln(\frac{2}{7})$ is the only vertical asymptote

(9)

- (18) 3. a) Write the definition of derivative as a limit and *use this definition* to find the derivative of $F(x) = x - x^2$.

- b) Use your answer to a) to find the equation of a line tangent to $y = x - x^2$ at the point where $x = 0$.

- c) Sketch $y = x - x^2$ and the line found in b) on the axes given.



$$\begin{aligned}
 \text{a) } F'(x) &= \lim_{\Delta x \rightarrow 0} \frac{F(x + \Delta x) - F(x)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{(x + \Delta x) - (x + \Delta x)^2 - (x - x^2)}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{x + \Delta x - x^2 - 2x\Delta x - (\Delta x)^2 - x + x^2}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{\Delta x - 2x\Delta x - (\Delta x)^2}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{1 - 2x - \Delta x}{1} \\
 &= 1 - 2x
 \end{aligned}$$

(b) $F'(0) = 1$ and $F(0) = 0$ so ~~the~~ an equation of the tangent line at $x=0$ is $y - F(0) = F'(0)(x - 0)$
which is $y = x$

(c) See above

- (14) 4. Find $\frac{dy}{dx}$ for each of the following:

a) $y = \frac{2x^2 + 5}{5x^3 + 1}$ b) $y = (4x + 3)\sqrt{x^3 + 7}$ c) $xy^3 = \cos(7x + 5y)$

a) Use the quotient rule, getting

$$\frac{dy}{dx} = \frac{(4x)(5x^3 + 1) - (2x^2 + 5)(15x^2)}{(5x^3 + 1)^2}$$

4(b) Use the product rule, getting:

$$\frac{dy}{dx} = 4\sqrt{x^3+7} + (4x+3)\left(\frac{1}{2}\right)\left(\frac{1}{\sqrt{x^3+7}}\right)(3x^2)$$

(c) Since $xy^3 = \cos(7x+5y)$

$$y^3 + 3xy^2 \frac{dy}{dx} = (-\sin(7x+5y))(7+5\frac{dy}{dx})$$

Thus $y^3 + 3xy^2 \frac{dy}{dx} = -7\sin(7x+5y) - 5\sin(7x+5y) \frac{dy}{dx}$

so $(3xy^2 + 5\sin(7x+5y)) \frac{dy}{dx} = -y^3 - 7\sin(7x+5y)$

Hence $\frac{dy}{dx} = \frac{-y^3 - 7\sin(7x+5y)}{3xy^2 + 5\sin(7x+5y)}$

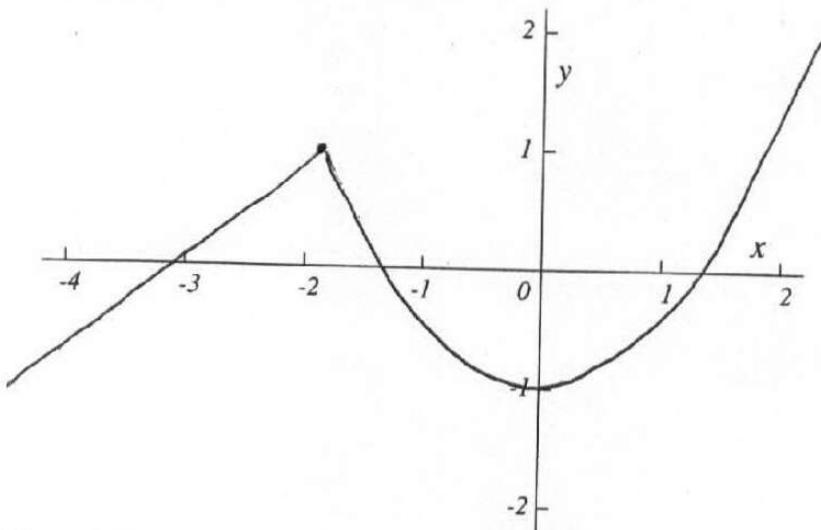
(16) 5. In this problem

$$W(x) = \begin{cases} x+3 & \text{if } x \leq -2 \\ \frac{1}{2}x^2 + A & \text{if } -2 < x \end{cases}$$

where A is a constant to be determined in part a).

a) Find A so that the function is continuous for all values of x .

b) Sketch a graph of $y = W(x)$ for $-4 \leq x \leq 2$ using the value of A found in a) on the axes given.



c) Is $W(x)$ differentiable at $x = -2$ using the value of A you have found in part a)? Explain your answer briefly.

(11)

$$\#5(a) \lim_{x \rightarrow -2^-} w(x) = \lim_{x \rightarrow -2^-} x+3 = -2+3=1$$

$$\lim_{x \rightarrow -2^+} w(x) = \lim_{x \rightarrow -2^+} \frac{1}{2}x^2 + A = A+2.$$

Thus $\lim_{x \rightarrow -2} w(x)$ exists if and only if $A=-1$.

In this case $\lim_{x \rightarrow -2} w(x) = 1 = w(1)$ so $w(x)$ is continuous at all other x (since any polynomial function is continuous).

(b) See graph

(c) $w(x)$ is not continuous at $x=-2$.

~~$$\lim_{x \rightarrow -2} w(x) = \lim_{\Delta x \rightarrow 0^-} \frac{w(-2+\Delta x) - w(-2)}{\Delta x}$$~~

$$= \lim_{\Delta x \rightarrow 0^-} \frac{-2 + \Delta x + 3 - (-2+3)}{\Delta x} = \lim_{\Delta x \rightarrow 0^-} \frac{\Delta x}{\Delta x} = 1$$

$$\lim_{\Delta x \rightarrow 0^+} \frac{w(-2+\Delta x) - w(-2)}{\Delta x} = \lim_{\Delta x \rightarrow 0^+} \frac{\left(\frac{1}{2}(-2+\Delta x)^2 - 1\right) - 1}{\Delta x}$$

~~$$= \lim_{\Delta x \rightarrow 0^+} \frac{\left(\frac{1}{2}(4 - 4\Delta x + (\Delta x)^2) - 2\right) - 2}{\Delta x} = \lim_{\Delta x \rightarrow 0^+}$$~~

$$= \lim_{\Delta x \rightarrow 0^+} \frac{\frac{1}{2}(4 - 4\Delta x + (\Delta x)^2) - 2}{\Delta x} = -4$$

Since these limits are not equal, $w(x)$ is not differentiable at $x=-2$.

(18) 6. Suppose that $N(x) = 5x^3 - 3x^5$.

- Compute $N'(x)$ and $N''(x)$. Where are each of these functions equal to 0?
- Find all relative maximum and minimum values of $N(x)$. Briefly explain your answers using calculus.
- Find all points of inflection of $N(x)$. Briefly explain your answers using calculus.

a) $N'(x) = 15x^2 - 15x^4 = 15x^2(1-x^2) = 15x^2(1-x)(1+x)$

Thus $N'(x) = 0$ at $x = -1, 0, 1$

$$N''(x) = 30x - 60x^3 = 30x(1-2x^2) = 30x(1-\sqrt{2}x)(1+\sqrt{2}x)$$

Thus $N''(x) = 0$ at $x = -\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}$

(b) $N'(x) < 0$ if $x < -1$

$$N'(x) > 0 \quad \text{if} \quad -1 < x < 0$$

$$N'(x) > 0 \quad \text{if} \quad 0 < x < 1$$

$$N'(x) < 0 \quad \text{if} \quad x > 1$$

Thus there is a relative minimum at $x = -1$:
The point is $(-1, -2)$



There is neither a relative minimum nor
a relative maximum at $x = 0$



There is a relative maximum at $x = 1$

The point is $(1, 2)$



(c) $N''(x) > 0$ if $x < -\frac{1}{\sqrt{2}}$

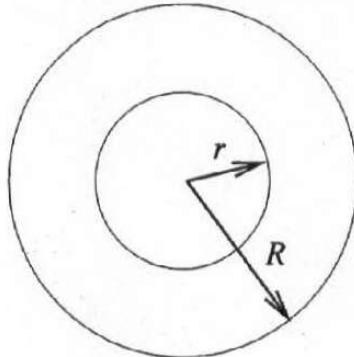
$$N''(x) < 0 \quad \text{if} \quad -\frac{1}{\sqrt{2}} < x < 0$$

$$N''(x) > 0 \quad \text{if} \quad 0 < x < \frac{1}{\sqrt{2}}$$

$$N''(x) < 0 \quad \text{if} \quad \frac{1}{\sqrt{2}} < x$$

Thus there are points of inflection at
 $x = -\frac{1}{\sqrt{2}}, x = 0, x = \frac{1}{\sqrt{2}}$ (for the concavity
changes at each of
these points).

- (12) 7. Two circles have the same center. The inner circle has radius r which is increasing at the rate of 3 inches per second. The outer circle has radius R which is increasing at the rate of 2 inches per second. Suppose A is the area of the region *between* the circles. At a certain time, r is 7 inches and R is 10 inches. What is A at that time? How fast is A changing at that time? Is A increasing or decreasing at that time?



$$A = \pi R^2 - \pi r^2 \text{ so when } R=10, r=7 \text{ we have}$$

$$A = \pi (100 - 49) = 51\pi.$$

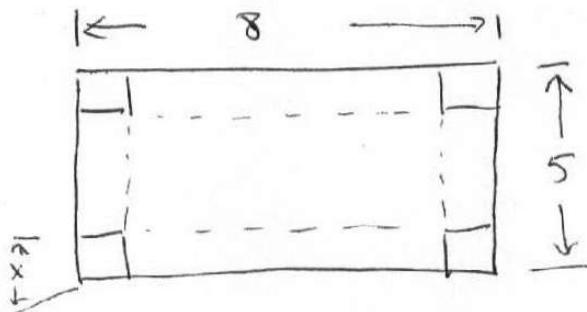
$$\frac{dA}{dt} = 2\pi R \frac{dR}{dt} - 2\pi r \frac{dr}{dt}. \text{ Thus when } R=10, r=7$$

(since we are told $\frac{dR}{dt} = 2$, $\frac{dr}{dt} = 3$) we have

$$\frac{dA}{dt} = 2\pi ((10)(2) - (7)(3)) = -2\pi. \text{ The negative value means that } A \text{ is decreasing.}$$

- (16) 8. A box with an open top is to be made from rectangular sheet of cardboard 5 inches by 8 inches by cutting equal squares out of the four corners and bending up the resulting four flaps to make the sides of the box. Use calculus to find the largest volume of the box.

Be sure to explain briefly why your answer gives a maximum.



(14)

The bottom of the box has dimensions $8-2x$ and $5-2x$. Thus the volume of the box is $V(x) = (8-2x)(5-2x)x = 4x^3 - 26x^2 + 40x$.

Then $V'(x) = 12x^2 - 52x + 40 = 4(3x^2 - 13x + 10) = 4(3x-10)(x-1)$. Thus the critical numbers are $x = \frac{10}{3}$, $x=1$.

We must have $x \geq 0$ and $5-2x \geq 0$ (so $x \leq \frac{5}{2}$). Since the critical number $\frac{10}{3}$ is not in $[0, \frac{5}{2}]$ the only x values that can produce an absolute maximum are the endpoints 0 and $\frac{5}{2}$ and the critical number 1. Now

$V(0) = V(\frac{5}{2}) = 0$, $V(1) = 18$. Thus the largest possible volume is 18 (at $x=1$).