Math 351

Workshop #5 **October 3, 2007**

The first problem asks you to find all the ideals in the polynomial ring $F[x]$ where F is a field. It is parallel to the determination of all ideals in the ring of integers (parts (a) and (b) of the first workshop problem last week).

#1 Let F be a field.

(a) Show that, for any polynomial $f(x) \in F[x]$, the set $\{f(x)g(x) | g(x) \in F[x]\} = f(x)F[x]$ is an ideal in $F[x]$. We will denote this ideal by $(f(x))$.

(b) Let I be an ideal in $F[x]$. Let P denote the set of integers k such that I contains a polynomial of degree k. Show that if $P = \emptyset$, then $I = (0)$.

(c) Let I be an ideal in $F[x]$ and let P be as in the previous part. Assume $P \neq \emptyset$. Then P contains a smallest element, say n, and I contains a polynomial, say $f(x)$, of degree n. Show that $I = (f(x))$.

The next problem relates sums and intersections of ideals in the ring $\mathbf{Z}$ of integers to the greatest common divisor and the least common multiple. Recall that if I and J are ideals in any ring R, then $I + J$ (defined to be $\{x + y | x \in I, y \in J\}$) and $I \cap J$ are ideals in R. Recall also that, by the second isomorphism theorem, the quotient rings $(I + J)/I$ and $J/(I \cap J)$ are isomorphic. Finally, recall that if $n \in \mathbf{Z}$, then (n) denotes $n\mathbf{Z} = \{nk | k \in \mathbf{Z}\}$ and that every ideal in $\mathbf{Z}$ is equal to (n) for some n.

#2 (a) Let a, b be nonzero integers. Define a *common multiple* of a and b to be an integer c such at a divides c and b divides c. Show that there is a smallest positive common multiple of a and b. This is called the *least common multiple* of a and b and is denoted by $[a, b]$.

(b) Let a, b be nonzero integers. Then $(a) \cap (b) = (n)$ for some positive integer n. Show that $n = [a, b]$.

(c) Let a, b be nonzero integers. Then $(a) + (b) = (m)$ for some positive integer m. Show that $m = (a, b)$ (the greatest common divisor of a and b).

(d) Let r, s be positive integers with $r|s$, say $s = qr$. Show that the quotient ring $(r)/(s)$ contains exactly q elements.

(e) Let a, b be positive integers. Apply the second isomorphism theorem to the ideals $(a) \cap (b)$ and $(a) + (b)$ in $\mathbf{Z}$ and use part (d) to conclude that $(a, b)[a, b] = ab$.